

FUZZY DIFFERENTIAL SUBORDINATIONS ASSOCIATED WITH A GENERALIZED DERIVATIVE OPERATOR

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RESEARCH ARTICLE

Abstract

The aim of the present paper is to establish several fuzzy differential subordinations by using a linear operator. Some interesting fuzzy consequences are also considered.

Keywords: multiplier transformations, differential subordination, fuzzy differential subordination, fuzzy best dominant.

INTRODUCTION

In 1965 Lotfi Zadeh introduced the special concept of fuzzy set in the work [19]. The new notion of fuzzy subordination was defined and studied recently in the papers [13-55]. This theory was developed in order to extend the classical differential subordination theory introduced and studied by S.S. Miller and P.T. Mocanu in [10]. In 2002 was studied the dual notion, of differential superordination [5]. See also interesting papers [3] and [11].

Since then, numerous researchers studied different properties [16] of differential operators involving fuzzy differential subordinations and superordinations: Wanas operator [2], [18], generalized Noor-Sălăgean operator [12], Sălăgean and Ruscheweyh operators [1] or other certain linear operator [9].

In this paper, fuzzy differential subordinations is added to the previous studies associated with a linear operator.

Denote by U the open unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let \mathcal{H} be the class of analytic functions in U and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ let $\mathcal{H}[a, n]$ be the subclass of \mathcal{H} consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, z \in U.$$

Let $\mathcal{A}(p, n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N} := \{1, 2, 3, \dots\})$$

which are analytic in the open unit disc. In particular, we set

$$\mathcal{A}(p, 1) := \mathcal{A}, \text{ and } \mathcal{A}(1, 1) := \mathcal{A} = \mathcal{A}_1.$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \dots\}$$

with $\mathcal{A}_1 = \mathcal{A}$.

In order to use the concept of fuzzy differential subordination, the following notions are necessary.

Definition 1.1. [19] A function $F : X \rightarrow [0, 1]$ is named fuzzy subset, where X is a non-empty set. Another definition will be the next one: A pair (A, F_A) with $F_A : X \rightarrow [0, 1]$

$$A = \{x \in X : 0 < F_A(x) \leq 1\} = \sup(A, F_A)$$

is named a fuzzy subset of X . Set A represents the support of the fuzzy set (A, F_A) . Also F_A is named the membership function of the fuzzy set (A, F_A) . One can also denote $A = \sup(A, F_A)$.

Remark 1.1. [15] Let be the inclusion relation $A \subset X$. Then we have

$$F_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The real number 0, for a fuzzy subset, is the smallest membership degree of $x \in X$ to A . Likewise, the real number 1 is the biggest membership degree of $x \in X$ to A .

The entire set X is associated with $F_X(x) = 1, x \in X$ and the empty set $\emptyset \subset X$ is associated with $F_\emptyset(x) = 0, x \in X$.

Definition 1.2 [13] Consider two functions $f, g \in H(D), D \subset \mathbb{C}, z_0 \in D$, being a fixed point. We say that the function f is fuzzy subordinate to g and written as

$$f \prec_F g \text{ or } f(x) \prec_F g(x), z \in D,$$

If the following relations are verified:

1. $f(z_0) = g(z_0)$;
2. $F_{f(D)}f(z) \leq F_{g(D)}g(z), z \in D$.

Definition 1.3 [14] Consider function $\Psi: \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let h be an univalent function in U satisfying

$$\Psi(a, 0, 0) = h(0) = a$$

We say that p is named a fuzzy solution of the fuzzy differential subordination if p is an analytic function in U , such that $p(0) = a$ and verifies the next (second-order) fuzzy differential subordination:

$$(1.1) \quad F_{\Psi(\mathbb{C}^3 \times U)} \Psi(p(z), zp'(z), z^2 p''(z); z) \leq F_{h(U)} h(z), z \in U$$

For all p satisfying (1.1) the univalent function q is named fuzzy dominant of the fuzzy solution for the fuzzy differential subordination, or a fuzzy dominant, if

$$F_{p(U)} p(z) \leq F_{q(U)} q(z), z \in U$$

A fuzzy dominant \tilde{q} which verifies

$$F_{\tilde{q}(U)} \tilde{q}(z) \leq F_{q(U)} q(z), z \in U,$$

for all fuzzy dominants q of (1.1) represents the fuzzy best dominant of (1.1).

MATERIAL AND METHOD

We begin our investigation by recalling here a generalized differential operator defined in [6].

Definition 2.1. [6] Let $f \in \mathcal{A}(p, n)$. For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \in \mathbb{R}, \lambda \geq 0, l \geq 0$, we define the

multiplier transformations $I_p^m(\lambda, l)$ on $A(p, n)$ by the following infinite series

$$(2.1) \quad I_p^m(\lambda, l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l} \right]^m a_k z^k.$$

It follows from (2.1) that

$$(2.2) \quad (p+l)I_p^{m+1}(\lambda, l)f(z) = [p(1-\lambda) + l]I_p^m(\lambda, l)f(z) + \lambda z(I_p^m(\lambda, l)f(z))'$$

Remark 2.1. For $p=1, l=0, \lambda \geq 0$, the operator $I_1^m(\lambda, 0) \equiv D_\lambda^m$ was introduced and studied by Al-Oboudi [4] which reduces to the Sălăgean differential operator [17] for $\lambda = 1$. The operator $I_1^m(1, l) \equiv I_l^m$ was studied recently by Cho and Srivastava [7] and Cho and Kim [8].

In this paper, we will derive several fuzzy subordination results involving the operator $I_1^m(\lambda, l)$ denoted by $I^m(\lambda, l)$. In order to prove our main results, we also need the following result.

Lemma 2.1 [10] Let q be a convex function in $U, \alpha \in \mathbb{C}^*$ and suppose that

$$\psi(z) = q(z) + n z q'(z),$$

where $n \in \mathbb{N}^*$.

If the function $h \in H[q(0), n]$ and $\varphi: \mathbb{C}^2 \times U \rightarrow \mathbb{C}$

$$\varphi(h(z), zh'(z)) = h(z) + \alpha zh'(z)$$

is analytic in U , with $h(0) = q(0)$, then

$$F_{\varphi(\mathbb{C}^2 \times U)} [h(z) + \alpha zh'(z)] \leq F_{\psi(U)} \psi(z), z \in U$$

i.e.

$$h(z) + \alpha zh'(z) \prec_{\mathcal{F}} \psi(z)$$

implies

$$F_{h(U)} h(z) \leq F_{q(U)} q(z), z \in U$$

i.e.

$$h(z) \prec_{\mathcal{F}} q(z)$$

and q is fuzzy best $(q(0), n)$ dominant.

RESULTS AND DISCUSSIONS

Theorem 3.1. Let q be a convex function in U , with $q(0)=1, \gamma \in \mathbb{C}^*$.

If $f \in \mathcal{A}(1, n)$ satisfies the fuzzy differential subordination

$$(3.1) \quad F_{\Phi(U)} \Phi(z) \leq F_{q(U)} \left(q(z) + \frac{\gamma \lambda}{1+l} z q'(z) \right)$$

where

$$(3.2) \quad \Phi(z) = \frac{I^m(\lambda, l)f(z)}{z} + \gamma \left(I^{m+1}(\lambda, l)f(z) - I^m(\lambda, l)f(z) \right)$$

then

$$\frac{I^m(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} q(z)$$

and q is the fuzzy best dominant of (3.1).

Proof. We define the function

$$(3.3) \quad h(z) := \frac{I^m(\lambda, l)f(z)}{z}$$

Using the form (2.1) of the operator (3.1) we deduce that $h \in \mathcal{H}[l, n]$, $q(0) = h(0)$.

Differentiating (3.2) with respect to z and using the identity (2.2) in the resulting equation we have

$$\frac{zh'(z)}{h(z)} = \frac{1+l}{\lambda} \left(\frac{I^{m+1}(\lambda, l)}{I^m(\lambda, l)} - 1 \right).$$

Therefore, we obtain

$$\frac{I^m(\lambda, l)f(z)}{z} + \gamma \left(I^{m+1}(\lambda, l)f(z) - I^m(\lambda, l)f(z) \right) = h(z) + \frac{\gamma\lambda}{1+l} zh'(z).$$

The fuzzy differential subordination (3.1) from the hypothesis becomes

$$F_{h(U)} \left(h(z) + \frac{\gamma\lambda}{1+l} zh'(z) \right) \leq F_{q(U)} \left(q(z) + \frac{\gamma\lambda}{1+l} zq'(z) \right)$$

i.e.

$$h(z) + \frac{\gamma\lambda}{1+l} zh'(z) \prec_{\mathcal{F}} q(z) + \frac{\gamma\lambda}{1+l} zq'(z).$$

Further, we apply Lemma 2.1 with $\alpha = \frac{\gamma\lambda}{1+l}$ to obtain the conclusion of the Theorem 3.1 \square

If we consider $m = 0$ in Theorem 3.1 we obtain the following result.

Corollary 3.1 Let q be a convex function in U , with $q(0)=1, \gamma \in \mathbb{C}^*$.

If $f \in \mathcal{A}(1, n)$ satisfies the fuzzy differential subordination

$$(3.4) \quad (1-\gamma) \frac{f(z)}{z} + \gamma \frac{I^1(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} q(z) + \frac{\gamma\lambda}{1+l} zq'(z)$$

i.e.

$$F_{G(U)} G(z) \leq F_{q(U)} \left(q(z) + \frac{\gamma\lambda}{1+l} zq'(z) \right)$$

where

$$G(z) = (1-\gamma) \frac{f(z)}{z} + \gamma \frac{I^1(\lambda, l)f(z)}{z},$$

then

$$\frac{f(z)}{z} \prec_{\mathcal{F}} q(z)$$

and q is the fuzzy best dominant of (3.4).

We consider a particular convex function $q(z) = \frac{1+Az}{1+Bz}$ to give the following application to Theorem 3.1.

Corollary 3.2 Let $\gamma \in \mathbb{C}^*, A \neq B$ such that $-1 \leq B < A \leq 1$ and $\text{Re } \gamma > 0$.

If $f \in \mathcal{A}(1, n)$ satisfies the fuzzy differential subordination

$$(3.5) \quad (1-\gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} \frac{1+Az}{1+Bz} + \frac{\gamma\lambda(A-B)z}{1+l(1+Bz)^2}$$

i.e.

$$F_{\Phi(U)} \Phi(z) \leq F_{q(U)} \left(\frac{1+Az}{1+Bz} + \frac{\gamma\lambda(A-B)z}{1+l(1+Bz)^2} \right)$$

where $\Phi(z)$ is defined in (3.2)

then

$$\frac{I^m(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} \frac{1+Az}{1+Bz}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best dominant of (3.5).

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 3.1 we obtain the following corollary.

Corollary 3.3 Let $\gamma \in \mathbb{C}^*$ and $f \in \mathcal{A}(1, n)$ satisfies the fuzzy differential subordination

$$(3.6) \quad (1-\gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} \frac{1+z}{1-z} + \frac{\alpha\lambda}{1+l} \frac{2z}{(1-z)^2}$$

i.e.

$$F_{\Phi(U)} \Phi(z) \leq F_{q(U)} \left(\frac{1+z}{1-z} + \frac{\alpha\lambda}{1+l} \frac{2z}{(1-z)^2} \right)$$

where $\Phi(z)$ is defined in (3.2)

then

$$\frac{I^m(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} \frac{1+z}{1-z}$$

and $q(z) = \frac{1+z}{1-z}$ is the best dominant of (3.6).

REFERENCES

- [1] Alb Lupas, G.I. Oros, New Applications of Salagean and Ruscheweyh Operators for Obtaining Fuzzy Differential Subordinations, *Mathematics*, 9 (2021), 2000.
- [2] S. Altınkaya, A.K. Wanas, Some properties for fuzzy differential subordination defined by Wanas operator, *Earthline J. Math. Sci.*, 4 (2020), 51–62.
- [3] W.G. Atshan, K.O. Hussain, Fuzzy Differential Superordination, *Theory Appl. Math. Comput. Sci.*, 7 (2017), 27–38.
- [4] F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, *Int. J. Math. Math. Sci.* 27, 1429-1436, (2004).
- [5] T. Bulboacă, Classes of first order differential superordinations, *Demonstratio Math.* 35(2), 287-292, (2002).
- [6] A. Cătaş, On certain class of p-valent functions defined by a new multiplier transformations, *Proceedings Book of the International Symposium G.F.T.A., Istanbul Kultur University, Turkey*, 241-250, (2007).
- [7] N.E. Cho and H.M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, *Math. Comput. Modelling*, 37(1-2) (2003), 39-49.
- [8] N.E. Cho and T.H. Kim, Multiplier transformations and strongly close-to-convex functions, *Bull. Korean Math. Soc.*, 40(3) (2003), 399-410.
- [9] S.M. El-Deeb, A. Alb Lupas, Fuzzy differential subordinations associated with an integral operator, *An. Univ. Oradea Fasc. Mat.*, XXVII (2020), 133–140.
- [10] S.S. Miller, P.T. Mocanu, *Differential Subordinations. Theory and Applications*, Marcel Dekker, Inc., New York, Basel, 2000.
- [11] S.S. Miller, P.T. Mocanu, Briot-Bouquet differential superordinations and sandwich theorems, *J. Math. Anal. Appl.*, {bf 329}(2007), No. 1, 327-335.
- [12] K.I. Noor, M.A. Noor, Fuzzy Differential Subordination Involving Generalized Noor-Salagean Operator, *Inf. Sci. Lett.*, 11 (2022), 1905–1911.
- [13] Oros G.I, Oros Gh., The notion of subordination in fuzzy sets theory. *General Mathematics*, 2011; 19: 97-103.
- [14] Oros G.I, Oros Gh., Fuzzy differential subordination. *Acta Universitatis Apulensis*, 2012; 30: 55-64.
- [15] Oros G.I, Oros Gh., Dominants and best dominants in fuzzy differential subordinations. *Stud Univ Babeş - Bolyai Math.* 2012; 57: 239-248.
- [16] G.I. Oros, Univalence criteria for analytic functions obtained using fuzzy differential subordinations, *Turk. J. Math.*, 46 (2022), 1478–1491.
- [17] G.S. Sălăgean, Subclasses of univalent functions, *Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1* (Bucharest, 1981), 362-372, *Lecture Notes in Math.*, 1013, Springer, Berlin (1983).
- [18] A.K. Wanas, Fuzzy differential subordinations of analytic functions involving Wanas operator, *Ikonian J. Math.*, 2 (2020), 1–9
- [19] L. A. Zadeh, *Fuzzy Sets*, *Inf. Control*, 8 (1965), 338–353.