

A NOTE ON CERTAIN FUZZY DIFFERENTIAL SUBORDINATIONS

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RESEARCH ARTICLE

Abstract

The paper aims to present a survey on certain fuzzy differential subordinations, namely fuzzy differential superordinations, for analytic functions defined in the open unit disc. The new results are derived by considering a certain differential operator. Some interesting further fuzzy consequences are also considered.

Keywords: multiplier transformations, fuzzy differential subordinations, fuzzy differential superordinations, fuzzy best subordinant.

INTRODUCTION

S.S. Miller and P.T. Mocanu have been introduced in Geometric Function Theory of one complex variable functions, the admissible functions method, known as "the differential subordination method", as we can see in the works [10-12]. There are many important applications of this method. One of them allows to obtain new results in the domain and also to prove certain classical results in the field.

Lotfi Zadeh in 1965 has introduced the special concept of fuzzy set in the work [22]. In recent papers G.I. Oros and Gh. Oros developed a new direction of this set, approaching the notion in the context of Geometric Function Theory. Thus, the papers [16-18] presents the new concept of fuzzy differential subordination.

The notion of differential superordination was introduced by Miller and Mocanu [14] as a dual concept of differential subordination and was also developed in [13]. Also, this notion was studied by T. Bulboacă in [5].

The new notion of fuzzy differential superordination was introduced in 2017 in [4]. Since then, numerous researches studied different properties of differential operators involving fuzzy differential subordination and fuzzy differential superordination [19], Wanas operator [3], [21], generalized Noor-Sălăgean

operator [15], Sălăgean and Ruscheweyh operators [2] or other certain linear operator [9].

In this paper, fuzzy differential superordinations is added to the previous studies associated with a linear operator.

Denote by U the open unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let \mathcal{H} be the class of analytic functions in U and for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ let $\mathcal{H}_{[a,n]}$ be the subclass of \mathcal{H} consisting of functions of the form $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots$, $z \in U$.

Let $\mathcal{A}(p,n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \quad (p, n \in \mathbb{N} = \{1, 2, 3, \dots\})$$

which are analytic in the open unit disc. In particular, we set $\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$. Let $\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \dots\}$ with $\mathcal{A}_1 = \mathcal{A}$.

We denote by Q the set of functions f that are analytic and injective on $\bar{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty\}$$

and are such that $f(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the term of fuzzy differential superordination, we review here some definitions.

Definition 1.1. [22] A function $F : X \rightarrow [0,1]$ is named fuzzy subset, where X is a non-empty set. Another definition will be the next one: A pair (A, F_A) with $F_A : X \rightarrow [0,1]$

$A = \{x \in X : 0 < F_A(x) \leq 1\}$ is named a fuzzy subset of X . Set A represents the support of the fuzzy set (A, F_A) . Also F_A is named the membership function of the fuzzy set (A, F_A) . One can also denote $A = \text{sup}(A, F_A)$.

Remark 1.1. [18] Let be the inclusion relation $A \subset X$. Then we have

$$F_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

The real number 0, for a fuzzy subset, is the smallest membership degree of $x \in X$ to A . Likewise, the real number 1 is the biggest membership degree of $x \in X$ to A .

The entire set X is associated with $F_X(x) = 1, x \in X$ and the empty set $\emptyset \subset X$ is associated with $F_\emptyset(x) = 0, x \in X$.

Definition 1.2 [16] Consider two functions $f, g \in H(D), D \subset \mathbb{C}, z_0 \in D$ being a fixed point. We say that the function f is fuzzy subordinate to g and written as

$$f \prec_F g \text{ or } f(x) \prec_F g(x), z \in D$$

If the following relations are verified:

1. $f(z_0) = g(z_0)$;
2. $F_{f(D)}f(z) \leq F_{g(D)}g(z), z \in D$.

Definition 1.3. [4] Let $\phi : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let h be analytic in U . If the functions p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent in U and satisfy the (second-order) fuzzy differential superordination

$$(1.1) \quad F_{h(U)}h(z) \leq F_{\phi(\mathbb{C}^3 \times U)}\left(\Phi\left(p(z), zp'(z), z^2p''(z); z\right)\right)$$

i.e.

$$h(z) \prec_F \Phi\left(p(z), zp'(z), z^2p''(z); z\right)$$

then p is called a fuzzy solution of the fuzzy differential superordination. An analytic function q is called fuzzy subordinate of the fuzzy differential superordination, or more simply a fuzzy subordination if $q(z) \prec_F p(z), z \in U$, for all p satisfying (1.1). A univalent fuzzy

subordination \tilde{q} that satisfies $\tilde{q} \prec_F q$ for all fuzzy subordinate q of (1.1) is said to be the fuzzy best subordinate of (1.1). Note that the fuzzy best subordinator is unique to a rotation of U .

MATERIAL AND METHOD

We begin our investigation by recalling here a generalized differential operator defined in [6].

Definition 2.1. [6] Let $f \in \mathcal{A}(p, n)$. For $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \lambda \in \mathbb{R}, \lambda \geq 0, l \geq 0$, we define the multiplier transformations $I_p^m(\lambda, l)$ on $\mathcal{A}(p, n)$ by the following infinite series

$$(2.1) \quad I_p^m(\lambda, l)f(z) := z^p + \sum_{k=p+n}^{\infty} \left[\frac{p+\lambda(k-p)+l}{p+l} \right]^m a_k z^k.$$

It follows from (2.1) that

$$(2.2) \quad (p+l)I_p^{m+1}(\lambda, l)f(z) = [p(1-\lambda) + l]I_p^m(\lambda, l)f(z) + \lambda z(I_p^m(\lambda, l)f(z))'$$

Remark 2.1. For $p = 1, l = 0, \lambda \geq 0$, the operator $I_1^m(\lambda, 0) \equiv D_\lambda^m$ was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [20] for $\lambda = 1$. The operator $I_1^m(1, l) \equiv I_l^m$ was studied recently by Cho and Srivastava [7] and Cho and Kim [8].

In this paper, we will derive several fuzzy superordination results involving the operator $I_1^m(\lambda, l)$ denoted by $I^m(\lambda, l)$. In order to prove our main results, we also need the following result.

Lemma 2.1 [4] Let q be convex in U and let p be defined by

$$p(z) = q(z) + \frac{1}{\alpha} zq'(z), \text{ with } \text{Re } \alpha > 0.$$

If $h \in \mathcal{H}[a, n] \cap Q$, the function $h(z) + \frac{1}{\alpha} zh'(z)$

is univalent in U and

$$q(z) + \frac{1}{\alpha} zq'(z) \prec_F h(z) + \frac{1}{\alpha} zh'(z),$$

i.e.

$$F_{p(U)}p(z) \leq F_{h(\mathbb{C}^2 \times U)}[h(z) + \frac{1}{\alpha} zh'(z)], z \in U$$

then

$$q(z) \prec_F h(z)$$

i.e.

$$F_{q(U)}q(z) \leq F_{h(U)}h(z), z \in U$$

where

$$q(z) = \frac{\alpha}{z^\alpha} \int_0^z p(t)t^{\alpha-1} dt$$

and q is the fuzzy best subdominant.

We recall further the next the following fuzzy differential "sandwich theorem"

Theorem 2.2. [4] Let h_1 and h_2 be convex in U , with $h_1(z) = h_2(z) = a$. Let $\alpha \in \mathbb{C}^*$, $\text{Re } \alpha > 0$ and let the function q_i be defined by

$$q_i(z) = \frac{\alpha}{z^\alpha} \int_0^z p(t)t^{\alpha-1} dt$$

for $i=1,2$. If $p \in \mathcal{H}[a,1] \cap Q$ and $p(z) + \frac{1}{\alpha} zp'(z)$ is univalent, then

$$h_1(z) \prec_{\mathcal{F}} p(z) + \frac{1}{\alpha} zp'(z) \prec_{\mathcal{F}} h_2(z) \rightarrow q_1(z) \prec_{\mathcal{F}} p(z) \prec_{\mathcal{F}} q_2(z), z \in U.$$

The function, q_1 and q_2 are convex and they are respectively the fuzzy best subdominant and fuzzy best dominant.

The authors established earlier the following theorem.

Theorem 2.3. Let q be a convex function in U , with $q(0)=1, \gamma \in \mathbb{C}^*$.

If $f \in \mathcal{A}(1,n)$ satisfies the fuzzy differential subordination

$$(3.1) \quad F_{\Phi(U)}\Phi(z) \leq$$

$$F_{q(U)}\left(q(z) + \frac{\gamma\lambda}{1+l}zq'(z)\right)$$

where

$$(3.2) \quad \Phi(z) = \frac{I^m(\lambda,l)f(z)}{z} +$$

$$\frac{\gamma}{z}\left(I^{m+1}(\lambda,l)f(z) - I^m(\lambda,l)f(z)\right),$$

then

$$\frac{I^m(\lambda,l)f(z)}{z} \prec_{\mathcal{F}} q(z)$$

and q is the fuzzy best dominant of (3.1).

RESULTS AND DISCUSSIONS

Theorem 3.1. Let q be convex in U , with $q(0) = 1, \gamma \in \mathbb{C}^*, \text{Re } \gamma > 0$.

If $f \in \mathcal{A}(1,n)$ such that

$$\frac{I^m(\lambda,l)f(z)}{z} \in \mathcal{H}[q(0),n] \cap Q \text{ and}$$

$$\frac{I^m(\lambda,l)f(z)}{z} + \frac{\gamma}{z}\left(I^{m+1}(\lambda,l)f(z) - I^m(\lambda,l)f(z)\right)$$

is univalent in U and the following fuzzy differential subordinations holds

$$(3.1)$$

$$q(z) + \frac{\lambda\gamma}{1+l}zq'(z) \prec_{\mathcal{F}} \frac{I^m(\lambda,l)f(z)}{z} + \frac{\gamma}{z}\left(I^{m+1}(\lambda,l)f(z) - I^m(\lambda,l)f(z)\right),$$

then

$$q(z) \prec_{\mathcal{F}} \frac{I^m(\lambda,l)f(z)}{z}$$

and q is the fuzzy best subdominant of (3.1).

Proof. We define the function

$$(3.2) \quad h(z) := \frac{I^m(\lambda,l)f(z)}{z}.$$

Differentiating (3.2) with respect to z and using the identity (2.2) in the resulting equation we have

$$\frac{zh'(z)}{h(z)} = \frac{1+l}{\lambda} \left(\frac{I^{m+1}(\lambda,l)}{I^m(\lambda,l)} - 1 \right).$$

Therefore, one obtains

$$\frac{I^m(\lambda,l)f(z)}{z} + \frac{\gamma}{z}\left(I^{m+1}(\lambda,l)f(z) - I^m(\lambda,l)f(z)\right) = h(z) + \frac{\gamma\lambda}{1+l}zh'(z).$$

The subordination (3.1) becomes

$$q(z) + \frac{\gamma\lambda}{1+l}zq'(z) \prec_{\mathcal{F}} h(z) + \frac{\gamma\lambda}{1+l}zh'(z).$$

The conclusion of this theorem follows by applying Lemma 2.1., with $\alpha = \frac{1+l}{\gamma\lambda}$.

Taking $m = 0$ in Theorem 3.1 we obtain the following result.

Corollary 3.1. Let q be convex in U , with $q(0)=1, \gamma \in \mathbb{C}^*, \text{Re } \gamma > 0$.

If $f \in \mathcal{A}(1,n)$ such that $\frac{f(z)}{z} \in \mathcal{H}[1,n] \cap Q$ and

$$(1-\gamma)\frac{f(z)}{z} + \gamma\frac{I^1(\lambda,l)f(z)}{z}$$

is univalent in U and the following fuzzy differential superordination holds

$$q(z) + \frac{\gamma\lambda}{1+l}zq'(z) \prec_{\mathcal{F}} (1-\gamma)\frac{f(z)}{z} + \gamma\frac{I^1(\lambda,l)f(z)}{z},$$

then

$$q(z) \prec_{\mathcal{F}} \frac{f(z)}{z}$$

and q is the fuzzy best subdominant.

Corollary 3.2. Let $\gamma \in \mathbb{C}^*, \text{Re } \gamma > 0, A \neq B$ such that $-1 \leq B < A \leq 1$.

If $f \in \mathcal{A}(1,n)$ such that

$$\frac{I^m(\lambda, l)f(z)}{z} \in \mathcal{H}[1, n] \cap Q$$

and

$$(1 - \gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z}$$

is univalent in U and satisfies the fuzzy differential superordination

$$\frac{1+Az}{1+Bz} + \frac{\gamma\lambda}{1+l} \frac{(A-B)z}{(1+Bz)^2} \prec_{\mathcal{F}} (1 - \gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z}$$

then

$$\frac{1+Az}{1+Bz} \prec_{\mathcal{F}} \frac{I^m(\lambda, l)f(z)}{z}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the fuzzy best subdominant.

Corollary 3.3. Let $\gamma \in \mathbb{C}^*$, $\text{Re } \gamma > 0$. If $f \in \mathcal{A}(1, n)$ such that

$$\frac{I^m(\lambda, l)f(z)}{z} \in \mathcal{H}[1, n] \cap Q$$

and

$$(1 - \gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z}$$

is univalent in U and satisfies the fuzzy differential superordination

$$\frac{1+z}{1-z} + \frac{\gamma\lambda}{1+l} \frac{2z}{(1-z)^2} \prec_{\mathcal{F}} (1 - \gamma) \frac{I^m(\lambda, l)f(z)}{z} + \gamma \frac{I^{m+1}(\lambda, l)f(z)}{z}$$

then

$$\frac{1+z}{1-z} \prec_{\mathcal{F}} \frac{I^m(\lambda, l)f(z)}{z}$$

and $q(z) = \frac{1+z}{1-z}$ is the fuzzy best subdominant.

Theorem 3.2. Let q_1, q_2 be convex in U , with $q_1(0) = q_2(0) = 1, \gamma \in \mathbb{C}^*, \text{Re } \gamma > 0$.

If $f \in \mathcal{A}(1, n)$ such that $\frac{I^m(\lambda, l)f(z)}{z} \in \mathcal{H}[1, n] \cap Q$ and

$$\frac{I^m(\lambda, l)f(z)}{z} + \frac{\gamma}{z} \left(I^{m+1}(\lambda, l)f(z) - I^m(\lambda, l)f(z) \right)$$

is univalent in U and satisfies

$$q_1(z) + \frac{\gamma\lambda}{1+l} z q_1'(z) \prec_{\mathcal{F}} \frac{I^m(\lambda, l)f(z)}{z} + \frac{\gamma}{z} \left(I^{m+1}(\lambda, l)f(z) - I^m(\lambda, l)f(z) \right) \prec_{\mathcal{F}} q_2(z) + \frac{\gamma\lambda}{1+l} z q_2'(z),$$

then

$$q_1(z) \prec_{\mathcal{F}} \frac{I^m(\lambda, l)f(z)}{z} + \frac{\gamma}{z} \left(I^{m+1}(\lambda, l)f(z) - I^m(\lambda, l)f(z) \right) \prec_{\mathcal{F}} q_2(z)$$

and q_1, q_2 are the fuzzy best subdominant and the fuzzy best dominant respectively.

Considering the operator $\frac{f(z)}{z}$ we obtain the corresponding sandwich corollary.

Corollary 3.4. Let q_1, q_2 be convex in U , with $q_1(0) = q_2(0) = 1, \gamma \in \mathbb{C}^*, \text{Re } \gamma > 0$. If $f \in \mathcal{A}(1, n)$ such that $\frac{f(z)}{z} \in \mathcal{H}[1, n] \cap Q$ and

$$(1 - \gamma) \frac{f(z)}{z} + \gamma \frac{I^1(\lambda, l)f(z)}{z}$$

is univalent in U and satisfies the double fuzzy differential subordinations

$$q_1(z) + \frac{\gamma\lambda}{1+l} z q_1'(z) \prec_{\mathcal{F}} (1 - \gamma) \frac{f(z)}{z} + \gamma \frac{I_p^1(\lambda, l)f(z)}{z} \prec_{\mathcal{F}} q_2(z) + \frac{\gamma\lambda}{1+l} z q_2'(z),$$

then

$$q_1(z) \prec_{\mathcal{F}} \frac{f(z)}{z} \prec_{\mathcal{F}} q_2(z)$$

and q_1, q_2 are the fuzzy best subdominant and the fuzzy best dominant respectively.

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