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DIFFERENTIAL SUBORDINATION RESULTS FOR CERTAIN GENERALIZED DIFFERENTIAL OPERATOR

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Abstract
The purpose of the paper is to deduce certain differential subordination results by making use of a generalized differential operator.

Key words: analytic functions, generalized differential operator, differential subordination.

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INTRODUCTION

1.
In the first section we will recall some definitions and results used for the new obtained results.
Denote by U the open unit disc of the complex plane:
$U = \{z \in \mathbb{C} : |z| < 1\}$.
Let $\mathcal{H}$ be the class of analytic functions in U and for $a \in \mathbb{C}$ and n ∈ N let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form
$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U$.
Let $\mathcal{A}(p,n)$ denote the class of functions $f(z)$ normalized by

(1.1) $$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k,\ (p, n \in \mathbb{N} := \{1,2,3,\ldots\})$$

which are analytic in the open unit disc.
In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p \text{ and } \mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1.$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1} z^{n+1} + \ldots\}$$

with $\mathcal{A}_1 := \mathcal{A}$.

We denote by Q the set of functions$f$ that are analytic and injective on$\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z \to \zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Since we use the terms of subordination and superordination, we review here those definitions.

If$f$and $g$are analytic functions in U, then we say that function$f$ is subordinate to g or g is said to be superordinate to f, if there exists a function $w$analytic in U, with $w(0)=0$ and $|w(z)|<1$, and such that $f(z) = g(w(z))$. In such case we write $f \prec g$ or $f(z) \prec g(z)$.

If g is univalent, then $f \prec g$ if and only if $f(0)=g(0)$ and $f(U) \subset g(U)$.

Further, we will recall here a differential operator introduced earlier.

Let the function f be in the class $\mathcal{A}_n$. For $m, \beta \in N_0 = \{0, 1, 2, \dots\}$, $\lambda \geq 0$, $l \geq 0$,

we will use the following differential operator

(1.2) $$I^m(\lambda, \beta, l)f(z) := z + \sum_{k=n+1}^{\infty} \left[\frac{1 + \lambda(k-1)+l}{1+l}\right]^m C(\beta, k) a_k z^k$$

where

$$C(\beta, k) := \binom{k+\beta-1}{\beta} = \frac{(\beta+1)_{k-1}}{(k-1)!}$$

and

$$(a)_n := \begin{cases} 1, & n = 0 \\ a(a+1)\dots(a+n-1), & n \in N - \{0\} \end{cases}$$

isPochhamer symbol.

Using simple computation one obtains the next result.

MATERIAL AND METHOD

2.PRELIMINARY RESULTS

Proposition 1.1For$m, \beta \in N_0$, $\lambda \geq 0$, $l \geq 0$

(2.1)

$$(l + 1)I^{m+1}(\lambda, \beta, l)f(z) = (1 - \lambda + l)I^m(\lambda, \beta, l)f(z) + \lambda z(I^m(\lambda, \beta, l)f(z)'$$

and

(2.2)

$$z(I^m(\lambda, \beta, l)f(z))' = (1 + \beta)I^m(\lambda, \beta + 1, l)f(z) - \beta I^m(\lambda, \beta, l)f(z).$$

Remark 2.1 Special cases of this operator includes the Ruscheweyh derivative operator $I^0(1, \beta, 0)f(z) \equiv D_\beta$ defined in [7], the Sălăgeanderivative operator $I^m(1, 0, 0)f(z) \equiv D^m$, studied in [8], the generalized*Sălăgean operator* $I^m(\lambda, 0, 0) \equiv D_\lambda^m$introduced by Al-Oboudi in

[1], thegeneralized Ruscheweyhderivative operator $I^1(\lambda, \beta, 0)f(z) \equiv D_{\lambda,\beta}$introduced in [6], the operator $I^m(\lambda, \beta, 0) \equiv D^m_{\lambda,\beta}$introduced by K. Al-Shaqsiand M. Darus in [2], and finally the operator $I^m(\lambda, 0, l) \equiv I_1(m, \lambda, l)$introduced in [3].

To prove the main results we will need the following lemma.

Lemma 2.1(Miller and Mocanu [4]) Let q be a convex function in Uand let

$$h(z) = q(z) + n\alpha z q'(z)$$

where α > 0 and n is a positive integer.

If

$$p(z) = q(0) + p_n z^n + \cdots \in \mathcal{H}[q(0), n]$$

and

$$p(z) + \alpha z p'(z) \prec h(z)$$

then

$$p(z) \prec q(z)$$

and this result is sharp.

3. MAIN RESULTS

Theorem 3.1Let q(z) be a convex function, q(0) = 1, and let h be afunction such that

(3.1) $$h(z) = q(z) + n\lambda z q'(z), \lambda > 0.$$

If f $\in \mathcal{A}_n$ and verifies the differential subordination

(3.2) $$(I^{m+1}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' \prec h(z)$$

where

$$\widetilde{\Psi}(\alpha, f; z) = z\Psi(\alpha, f; z)$$

$$\Psi(\alpha, f; z) = (1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right),$$

then

(3.3) $$(I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' \prec q(z)$$

and the result is sharp.

Proof.By using the properties of the operator $I^m(\lambda, \beta, l)$, we have

(2.4) $$(l+1)I^{m+1}(\lambda, \beta, l)f(z) = (1-\lambda+l)I^m(\lambda, \beta, l)f(z) + \lambda z(I^m(\lambda, \beta, l)f(z))'.$$

If we denote by

(3.5) $$p(z) = (I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))'$$

where $p(z) = 1 + p_n z^n + \ldots, p(z) \in \mathcal{H}[1, n]$,

then,

after a short computation we get

(3.6) $$(I^{m+1}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' = p(z) + \lambda z p'(z), z \in U.$$

From (3.4), (3.5) and (3.2) we obtain

(3.7) $$p(z) + \lambda z p'(z) \prec q(z) + n\lambda z q'(z) \equiv h(z)$$

then,

by using Lemma 2.1 we get

$$p(z) \prec q(z)$$

or

$$(I^{m}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' \prec q(z), \quad z \in U,$$

and this result is sharp. □

Theorem 3.2Let q be a convex function with q(0) = 1 and let h be afunction of the form

(3.8) $$h(z) = q(z) + nzq'(z), z \in U.$$

If f $\in \mathcal{A}_n$ verifies the differential subordination

(3.9) $$(I^{m}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' \prec h(z), \quad z \in U,$$

then

(3.10) $$(I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))/z \prec q(z)$$

and this result is sharp.

Proof.If we let

$$p(z) = (I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))/z, \ z \in U,$$

then we obtain

$$(I(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' = p(z) + zp'(z), \quad z \in U.$$

The subordination (3.9) becomes

$$p(z) + zp'(z) \prec q(z) + nzq'(z)$$

and from Lemma 2.1 we have (3.10). The result is sharp. □

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