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DIFFERENTIAL SANDWICH RESULTS FOR P-VALENT FUNCTIONS

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Abstract
In this paper we derive some applications of first order differential subordination and results involving a generalized multiplier transformations. Applying the techniques of differential subordination and superordination we also establish a differential sandwich-type theorem.

Key words: multiplier transformations, differential subordination, differential superordination.

INTRODUCTION

1.
Denote by U the open unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : |z| < 1\}.$$

Let $\mathcal{H}$ be the class of analytic functions in U and for $a \in \mathbb{C}$ and $n \in N$ let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots, \ z \in U.$$

Let $\mathcal{A}(p,n)$ denote the class of functions $f(z)$ normalized by

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k, \ (p, n \in \mathbb{N} := \{1,2,3,\dots\})$$

which are analytic in the open unit disc. In particular, we set

$$\mathcal{A}(p,1) := \mathcal{A}_p \text{ and } \mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1.$$

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), \ f(z) = z + a_{n+1} z^{n+1} + \dots\}$$

with $\mathcal{A}_1 = \mathcal{A}$.
We denote by Q the set of functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where

$$E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.
Since we use the terms of subordination and superordination, we review here those definitions.

Let f, $F \in \mathcal{H}$. The function f is said to be subordinate to F or F is said to be superordinate to f, if there exists a function w analytic in U, with $w(0)=0$ and $|w(z)|<1$, and such that $f(z) = F(w(z))$. In such case we write $f \prec F$ or $f(z) \prec F(z)$.
If F is univalent, then $f \prec F$ if and only if $f(0)=F(0)$ and $f(U) \subset F(U)$.
Since most of the functions considered in this paper and conditions on them are defined uniformly in the unit disk U, we shall omit the requirement " $z \in U$ ".
Let $\psi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let h be univalent in U and $q \in Q$. In [6] the authors considered the problem of determining conditions on admissible function ψ such that

$$\psi\,(p(z), zp'(z), z^2p''(z); z) \prec h(z) \tag{1.2}$$

implies $p(z) \prec q(z)$, for all functions $p \in \mathcal{H}[a,n]$ that satisfy the differential subordination (1.2).
Moreover, they found conditions so that the function q is the "smallest" function with this property, called the best dominant of the subordination (1.2).
Let $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, let $h \in \mathcal{H}$ and $q \in \mathcal{H}[a,n]$. Recently, in [7] the authors studied the dual problem and determined conditions on φ such that

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z) \tag{1.3}$$

implies $p(z) \prec q(z)$, for all functions $p \in Q$ that satisfy the above differential superordination.
Moreover, they found conditions so that the function q is the "largest" function with this property, called the best subordinant of the superordination (1.3).
For two functions

$$f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k \text{ and } g(z) = z^p + \sum_{k=p+n}^{\infty} b_k z^k,$$

the Hadamard product of f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=p+n}^{\infty} a_k b_k z^k.$$

MATERIAL AND METHOD

2. PRELIMINARY RESULTS

We begin our investigation by recalling here a generalized differential operator defined in [3].

Definition 2.1. [3] Let $f \in \mathcal{A}(p,n)$. For $m \in \mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\lambda \in \mathbb{R}$, $\lambda \geq 0$, $l \geq 0$, we define the multiplier trasformations $I_p^m(\lambda, l)$ on $\mathcal{A}(p,n)$ by the following infinite series

(2.1) $$I_p^m(\lambda, l)f(z) := z^p + \sum_{k=p+n}^{\infty}\left[\frac{p+\lambda(k-p)+l}{p+l}\right]^m a_k z^k.$$

It follows from (2.1) that

(2.2)
$$(p+l)I_p^{m+1}(\lambda, l)f(z) = [p(1-\lambda)+l]I_p^m(\lambda, l)f(z) + \lambda z(I_p^m(\lambda, l)f(z))'.$$

Remark 2.1 For $p = 1$, $l = 0$, $\lambda \geq 0$, the operator $I_1^m(\lambda, 0) \equiv D_\lambda^m$ was introduced and studied by Al-Oboudi [1] which reduces to the Sălăgean differential operator [8] for $\lambda = 1$. The operator $I_1^m(1, l) \equiv I_l^m$ was studied recently by Cho and Srivastava [4] and Cho and Kim [5].

In this paper, we will derive several subordination and superordination results involving the operator $I_p^m(\lambda, l)$. In order to prove our main results, we also need the following result.

Lemma 2.1 [2] Let q be convex in U, $q(0) = a$ and $\gamma \in \mathbb{C}$, Re $\gamma > 0$.

If $h \in \mathcal{H}[a,1] \cap Q$, the function $h(z) + \gamma z h'(z)$ is univalent in U and

$$q(z) + \gamma z q'(z) \prec h(z) + \gamma z h'(z),$$

then

$$q(z) \prec h(z)$$

and q is the best subordinant.

The authors established earlier the following theorem

Theorem 2.1. Let q be univalent in U, with $q(0)=1$, $\alpha \in \mathbb{C}^*$, m, $\beta \in \mathbb{N}_0=\{0, 1, 2, ...\}$ and suppose

$$\mathrm{Re}\left[1+\frac{zq''(z)}{q'(z)}\right] > \max\left\{0, -\frac{p+l}{\lambda} Re\frac{1}{\alpha}\right\}.$$

If $f \in \mathcal{A}(p,n)$ satisfies the subordination

(2.3)
$$\frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\alpha}{z^p}\left(I_p^{m+1}(\lambda, l)f(z) - I_p^m(\lambda, l)f(z)\right) \prec q(z) + \frac{\alpha\lambda}{p+l} z q'(z),$$

then

$$\frac{I_p^m(\lambda,l)f(z)}{z^p} \prec q(z)$$
and q is the best dominant of (2.3).

3. MAIN RESULTS

Theorem 3.1. Let q be convex in U, with $q(0) = 1$, $\alpha \in \mathbb{C}$, Re $\alpha > 0$. If $f \in \mathcal{A}(p,n)$ such that $\frac{I_p^m(\lambda,l)f(z)}{z^p} \in \mathcal{H}[q(0),1] \cap Q$ and
$$\frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right)$$
is univalent in U and the following superordinations holds

(3.1)
$$q(z) + \frac{\alpha\lambda}{p+l}zq'(z) \prec \frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right),$$
then
$$q(z) \prec \frac{I_p^m(\lambda,l)f(z)}{z^p}$$
and q is the best subordinant of (3.1).

Proof. We define the function

$$h(z) \coloneqq \frac{I_p^m(\lambda,l)f(z)}{z^p}. \tag{3.2}$$

Differentiating (3.2) with respect to z and using the identity (2.2) in the resulting equation we have
$$\frac{zh'(z)}{h(z)} = \frac{1}{\lambda}\left\{(p+l)\frac{I_p^{m+1}(\lambda,l)}{I_p^m(\lambda,l)} - [p(1-\lambda) + l + \lambda p]\right\}.$$
Therefore, we obtains
$$\frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right) = h(z) + \frac{\alpha\lambda}{p+l}zh'(z).$$

The subordination (3.1) becomes
$$q(z) + \frac{\alpha}{\beta+1}zq'(z) \prec h(z) + \frac{\alpha}{\beta+1}zh'(z).$$
The conclusion of this theorem follows by applying Lemma 2.1., with $\frac{\alpha}{\beta+1} = \gamma$.

Taking m = 0 in Theorem 3.1 we obtain the following result.

Corollary 3.1 Let q be convex in U, with $q(0)=1$, $\alpha\in\mathbb{C}$, with Re $\alpha > 0$.

If $f \in \mathcal{A}(p,n)$ such that $\frac{f(z)}{z^p} \in \mathcal{H}[1,1] \cap Q$ and
$$(1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{I_p^1(\lambda,l)f(z)}{z^p}$$
is univalent in U and the following superordinations holds
$$q(z) + \frac{\alpha\lambda}{p+l}zq'(z) \prec (1-\alpha)\frac{f(z)}{z^p} + \alpha\frac{I_p^1(\lambda,l)f(z)}{z^p},$$

then

$$q(z) \prec \frac{f(z)}{z^p}$$

and q is the best subordinant.

Corollary 3.2 Let $\alpha \in \mathbb{C}$, $A \neq B$ such that $-1 \leq B < A \leq 1$ and Re $\alpha > 0$. If $f \in \mathcal{A}(p,n)$ such that

$$\frac{I_p^m(\lambda,l)f(z)}{z^p} \in \mathcal{H}[1,1] \cap Q$$

and

$$(1-\alpha)\frac{I_p^m(\lambda,l)f(z)}{z^p} + \alpha\frac{I_p^{m+1}(\lambda,l)f(z)}{z^p}$$

is univalent in U and satisfies the superordination

$$\frac{1+Az}{1+Bz} + \frac{\alpha\lambda}{p+l}\frac{(A-B)z}{(1+Bz)^2} \prec (1-\alpha)\frac{I_p^m(\lambda,l)f(z)}{z^p} + \alpha\frac{I_p^{m+1}(\lambda,l)f(z)}{z^p}$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{I_p^m(\lambda,l)f(z)}{z^p}$$

and $q(z) = \frac{1+Az}{1+Bz}$ is the best subordinant.

Corollary 3.3 Let $\alpha \in \mathbb{C}$ and Re $\alpha > 0$. If $f \in \mathcal{A}(p,n)$ such that

$$\frac{I_p^m(\lambda,l)f(z)}{z^p} \in \mathcal{H}[1,1] \cap Q$$

and

$$(1-\alpha)\frac{I_p^m(\lambda,l)f(z)}{z^p} + \alpha\frac{I_p^{m+1}(\lambda,l)f(z)}{z^p}$$

is univalent in U and satisfies the superordination

$$\frac{1+z}{1-z} + \frac{\alpha\lambda}{p+l}\frac{2z}{(1-z)^2} \prec (1-\alpha)\frac{I_p^m(\lambda,l)f(z)}{z^p} + \alpha\frac{I_p^{m+1}(\lambda,l)f(z)}{z^p}$$

then

$$\frac{1+z}{1-z} \prec \frac{I_p^m(\lambda,l)f(z)}{z^p}$$

and $q(z) = \frac{1+z}{1-z}$ is the best subordinant.

Theorem 3.2. Let q_1, q_2 be convex in U, with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$, Re $\alpha > 0$ and q_2 satisfies the inequality

$$\text{Re}\left[1 + \frac{zq_2''(z)}{q_2'(z)}\right] > \max\left\{0, -\frac{\lambda}{p+l} Re\ \alpha\right\}.$$

If $f \in \mathcal{A}(p,n)$ such that $\frac{I_p^m(\lambda,l)f(z)}{z^p} \in \mathcal{H}[1,1] \cap Q$ and

$$\frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right)$$

is univalent in U and satisfies

$$q_1(z) + \frac{\alpha\lambda}{p+l} z q_1'(z) \prec \frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right) \prec$$
$$\prec q_2(z) + \frac{\alpha\lambda}{p+l} z q_2'(z),$$
then
$$q_1(z) \prec \frac{I_p^m(\lambda,l)f(z)}{z^p} + \frac{\propto}{z^p}\left(I_p^{m+1}(\lambda,l)f(z) - I_p^m(\lambda,l)f(z)\right) \prec q_2(z)$$
and q_1, q_2 are the best subordinant and the best dominant respectively.

Considering the operator $\frac{f(z)}{z^p}$ we obtain the coresponding sandwich corollary.

Corollary 3.4 Let q_1, q_2 be convex in U, with $q_1(0) = q_2(0) = 1$, $\alpha \in \mathbb{C}$, Re $\alpha > 0$ and q_2 satisfies the inequality
$$\text{Re}\left[1 + \frac{z q_2''(z)}{q_2'(z)}\right] > \max\left\{0, -\frac{\lambda}{p+l} Re\, \alpha\right\}.$$
If $f \in \mathcal{A}(p,n)$ such that $\frac{f(z)}{z^p} \in \mathcal{H}[1,1] \cap Q$ and
$$(1-\alpha)\frac{f(z)}{z^p} + \alpha \frac{I_p^1(\lambda,l)f(z)}{z^p}$$
is univalent in U and satisfies
$$q_1(z) + \frac{\alpha\lambda}{p+l} z q_1'(z) \prec (1-\alpha)\frac{f(z)}{z^p} + \alpha \frac{I_p^1(\lambda,l)f(z)}{z^p} \prec q_2(z) + \frac{\alpha\lambda}{p+l} z q_2'(z),$$
then
$$q_1(z) \prec \frac{f(z)}{z^p} \prec q_2(z)$$
and q_1, q_2 are the best subordinant and the best dominant respectively.

REFERENCES

1. F.M. Al-Oboudi, On univalent functions defined by a generalized Sălăgean operator, Int. J. Math. Math. Sci. 27, 1429-1436, (2004).
2. T. Bulboacă, Classes of first order differential superordinations, Demonstratio Math. 35(2), 287-292, (2002).
3. A.Cătaș, On certain class of p-valent functions defined by a new multiplier transformations, Proceedings Book of the International Symposium G.F.T.A., Istanbul Kultur University, Turkey, 241-250, (2007).
4. N.E. Cho and H.M. Srivastava, Argument estimates of certain analytic functions defined by a class of multiplier transformations, Math. Comput. Modelling, 37(1-2) (2003), 39-49.
5. N.E. Cho and T.H. Kim, Multiplier transformations and strongly close-to-convex functions, Bull. Korean Math. Soc., 40(3) (2003), 399-410.
6. S.S. Miller, P.T. Mocanu, Differential Subordinations. Theory and Applications, Marcel Dekker, Inc., New York, Basel, 2000.
7. S.S. Miller, P.T. Mocanu, Briot-Bouquet differential superordinations and sandwich theorems, J. Math. Anal. Appl., {\bf 329}(2007), No. 1, 327-335.
8. G.S.Sălăgean, Subclasses of univalent functions, Complex Analysis-Fifth Romanian-Finnish Seminar, Part 1 (Bucharest, 1981), 362-372, Lecture Notes in Math., 1013, Springer, Berlin (1983).