NUMERICAL METHOD FOR SOLVING NONLINEAR EQUATIONS FROM ASTRONOMY

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Abstract

In this paper we propose a numerical method for solving nonlinear equations from astronomy. In order to ensure the maximum number of exact digits after dot in the approximate solution we propose to apply multiple iterations to Newton's method. Proposed method is a gain, significantly reducing the number of iterations.

Key words: *nonlinear equations, the method of Newton, Kepler's equation, multiple iterations*

INTRODUCTION

In the year 2001 S. Sheppard and D.C. Jewitt discovered the moon Margaret of the planet Uranus. The observations that followed confirmed that it is a distinct celestial body and on 9 October 2003 it was included in the list as the 27^{th} moon of Uranus. The current eccentricity of the moon Margaret, in the year 2008, was 0.7979, temporarily giving it the most eccentric orbit of any moon in the solar system.

One of the methods for solving nonlinear equations is the Newton's method. A nonlinear equation to which one can apply the method of Newton is Kepler's equation from Astronomy. The math expression between polar coordinates of a celestial body and the time from a fixed point is provided by the equation of Kepler. It is to be noted that the equation of Kepler cannot be reversed from the point of view of simple functions to see where the planet will be at a fixed time.

Kepler's equation has been solved numerically using the Newton's method as can be seen in (Montenbruck O. et al., 2009), while in (Aaboe A., 2011) is used the method of successive approximations, (Colwell P., 1993), (Danby J.M.A., 1988), (Demidowitch B. et al., 1979), (Swerdlow N.M., 2000). It can be seen that the number of exact digits in these approximations could not be established with certitude when the Lipschitz constant of the involved function is greater than 0.5. Here we propose a refinement of Newton's method by multiple iterations such that to ensure the maximum number of exact digits in the approximate solution.

MATERIAL AND METHOD

Problem of nonlinear equations can be generically written as:

$$f(x) = 0 \ , x \in R \tag{1}$$

Such equations occur frequently in the analysis of systems in the field of Astronomy. Generally, it is impossible to calculate the solutions of nonlinear equations by a finite number of arithmetic operations. It requires an iterative method, that is a procedure that generates an infinite sequence of approximations $\{x_n , n \in N\}$ so that $\lim_{n->\infty} x_n = \alpha$, where α is a solution of the equation.

It is assumed that the function $f(x)$ is continuous on the interval $[a,b]$ and has a real solution $\alpha \in [a,b]$, with $f\,'\,(x)$ and $f\,''\,(x)$ being continuous and keep the sign.

Equation (1) is transformed in equivalent form:

$$x = \varphi(x) \tag{2}$$

Starting from the initial approximation x_0 for the solution α results the recurrent string of successive approximations: $x_{i+1} = \varphi(x_i), i = 0,1,2....$ If the string is convergent it results exist $\alpha = \lim_{n->\infty} x_n$. If $\varphi(x)$ is continuous it results that $\alpha = \varphi(\alpha)$ is the solution of the equation. Process is convergent only in the intervals where $|\varphi(x)^{'}| < 1$.

Theorem. Let the equation $x = \varphi(x)$, with the function $\varphi(x)$ defined and derivable on $[a,b]$. If inequality $|\varphi(x)^{'}| \leq \lambda < 1$ is satisfied for any $x \in [a,b]$, then the iteration string defined by the relation $x_{i+1} = \varphi(x_i), i = 0,1,2...$ converges to the solution of the equation, regardless of the initial value x_0.

Process converges more quickly to the solution α when λ is lower.

In Newton's method the equation (1) can be replaced with the equivalent equation:

$$x = x - \frac{f(x)}{f^{'}(x)} \tag{3}$$

that is, the iteration function is:

$$\varphi(x) = x - \frac{f(x)}{f^{'}(x)} \qquad (4)$$

Consequently, iterative process is: $x_{i+1} = x_i - \frac{f(x)}{f^{'}(x)}, i = 0,1,2,...$. Convergence criterion is $\delta_i \equiv \left|\frac{\Delta x_i}{x_{i+1}}\right| \le \varepsilon$ or $|\Delta x_i| \le \varepsilon |x_{i+1}|$. Solution's correction is $\Delta x_i = x_{i+1} - x_i = -\frac{f(x_i)}{f^{'}(x_i)}$.

In graphical interpretation from Figure 1 it is observed that starting from the initial approximation x_0 for α, the improved approximation x_{i+1} is obtained by tangent to $y = f(x)$ in $(x_i, f(x_i))$.

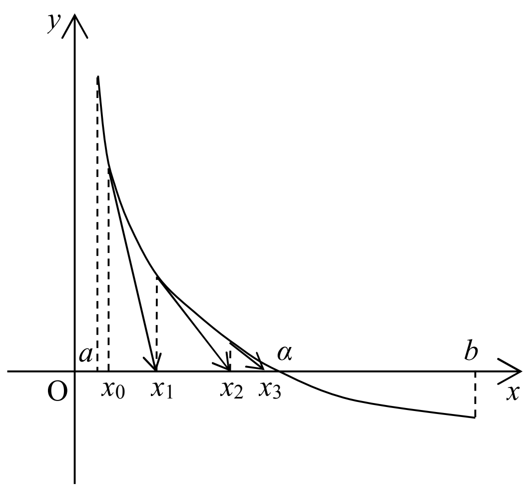

Fig. 1. Determining successive approximation in Newton's method.

In general, choosing starting value is a difficult problem. In practice, a value is chosen, and if after a fixed maximum number of iterations the desired precision has not been obtained, tested by one of the usual criteria, try a different start value. If the root is isolated in an interval $[a,b]$ and $f^{''}(x) \ne 0, x \in (a,b)$, a criterion of choice is $f(x_0) f^{''}(x_0) > 0$. Another criterion is: if f is convex or concave on the interval $[a,b]$, $f(a)f(b) < 0$ and the tangents in the ends intersect the axis Ox in the interval (a,b), it is possible to choose any value $x_0 \in [a,b]$ as starting value.

Each real solution of equation (4) is isolated in a specified interval by the method of roots separation generated by the technique of the Rolle's sequence when Newton's method is used, see (Demidowitch B. et al., 1979). Further, each such interval is shorted so that the first derivative φ' respects $|\varphi'(x)| < 1$ on the interval [a, b]. Thus, starting from any first value x_0 the recurring sequence of successive approximations converges to the solution of equation (4).

If $\alpha \in [a, b]$ is the exact solution of the equation (4), then $\lim_{n\to\infty} x_n = \alpha$ and the following a posteriori error estimate:

$$|\alpha - x_n| \le \frac{L}{1-L}|x_n - x_{n-1}|, \forall n \in N^* \tag{5}$$

where $L = \max\{|\varphi'(x)| : x \in [a, b]\}, L < 1$ is the Lipschitz constant of φ, see (Demidowitch B. et al., 1979). The math expression (5) provides a practical stopping criterion of the recurrent sequence, according to which for a fixed $\varepsilon > 0$ the sequence stops at the first $n \in N^*$ for which $|x_n - x_{n-1}| < \varepsilon$. Thus the precision in the estimation x_n of α is $|\alpha - x_n| \le \frac{L\varepsilon}{1-L}$.

If the Lipschitz constant $L \le 0.5$ the inequality $|x_n - x_{n-1}| < \varepsilon = 10^{-k}$ offers the insurance that the first k-1 digits after dot in the approximate solution are accurate. But if 0.5<L<1 the inequality $|x_n - x_{n-1}| < \varepsilon = 10^{-k}$ does not ensure the accuracy of the first k-1 digits after dot in the approximate solution.

As can be seen in the paper (Curila M. et al., 2013) the multiple iteration based on the iterative sequence

$$x_{i+1} = (\underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{p\,times})(x_i), i \in N \tag{6}$$

extends the interval (0 , 0.5] to $\left(0, \frac{1}{\sqrt[p]{2}}\right]$ for the Lipschitz constant L, because the a posteriori error estimate becomes:

$$|\alpha - x_n| \le \frac{L^p}{1-L^p} \cdot |x_n - x_{n-1}|, \forall n \in N^* \tag{7}$$

so that for $\varepsilon = 10^{-k}$ the first k-1 digits are exact in the estimation x_n. Consequently, by multiple iteration, the interval (0 , 0.5] is extended to the interval $\left(0, \frac{1}{\sqrt[p]{2}}\right]$ for the Lipschitz constant L, in order to ensure the maximum number of exact digits after dot in the approximate solution.

NUMERICAL EXPERIMENT

This is where the multiple iteration method based on Newton's method applied to numerical solving of Kepler's equation from Astronomy is being tested.

Let q be the mean anomaly (a parameterization of time) and x the eccentric anomaly (a parameterization of polar angle) of a celestial body orbiting on an ellipse with eccentricity E, then Kepler's equation is given by:

$$x = E \cdot \sin(x) + q \tag{8}$$

where $E \in (0,1]$ and q > 0, see (Aaboe A., 2011) and (Swerdlow N.M., 2000).

Taking E=1 and q=0.25, corresponding to a comet, the equation (8) becomes:

$$x = \sin(x) + 0.25,\ x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \tag{9}$$

As shown in the previous section, it follows $f(x) = x - \sin(x) - 0.25$.

In this case we use triple iteration that ensures the maximum number of exact digits after dot in the approximate solution. Taking $x_0 = \frac{\pi}{4}$ and $\varepsilon = 10^{-4}, 10^{-8}, 10^{-12}$ we obtain with this method the results presented in table 1:

Table 1

The method of triple iteration

ε	N	x_n	exact digits
10^{-4}	4	1.1712	1.171
10^{-8}	7	1.171348536	1.1713485
10^{-12}	11	1.171348543621	1.17134854362

With simple iteration the results are presented in table 2:

Table 2

Newton's method

ε	N	x_n	exact digits
10^{-4}	9	1.1712	1.171
10^{-8}	15	1.171348536	1.1713485
10^{-12}	21	1.171348543621	1.17134854362

In the case of moon Margaret of planet Uranus, the experimental results obtained for the approximation of Kepler's equation solution with the Newton's method and with the method of triple iteration are presented in table 3:

Table 3

Approximation of Kepler's equation solution for Margaret moon of Uranus planet

Newton's method	triple iteration
ε =1e-16	ε =1e-16
n=11	n=4
$x_0 \div x_{11}$	$x_0 \div x_4$
0.7853981633974483	0.7853981633974483
1.0112803788642934	1.4315761978365244
1.1329967434762813	1.6875781214952719
1.2985767849963402	1.6993557606481634
1.4315761978365244	1.6993557606481634
1.5578139610843792	
1.6866127654223918	
1.6875781214952719	
1.6984587086498526	
1.6988857086388278	
1.6993557606481634	
1.6993557606481634	

As can be viewed in these two examples, by using the Newton's method we need to repeat once the iterations when the sequence becomes stationary, in order to be sure about the number of iterations. So, our proposed method offers the certitude for the number of exact digits.

CONCLUSIONS

From a numerical point of view, the proposed method is a gain compared to the Newton's simple iteration method, significantly reducing the number of iterations.

The multiple iteration method improves the Newton's simple iteration method with regard to extending the interval for the Lipchitz constant from (0 , 0.5] to $\left(0, \frac{1}{\sqrt[p]{2}}\right]$, so as to ensure the maximum number of exact digits in the approximate solution of the nonlinear equation.

Although it requires the same conditions as Newton's simple iteration method, in addition to this, the multiple iteration method provides certainty about the number of exact digits of the approximate solution.

This numerical method used for the equation of Kepler can also be applied to other nonlinear equations occurring in various other applied areas.

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