# ON THE PRECISION OF THE SUCCESSIVE APPROXIMATIONS METHOD TESTED ON KEPLER'S EQUATION FROM ASTRONOMY

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***Abstract***

*In this paper we point out a method to ensure the same precision for the exact solution as the precision obtained by the last two iterations in the method of successive approximations. The method of double and multiple iteration could be an improvement of the method of successive approximations in the aim to extend the interval for the Lipchitz constant, such that the maximum number of exact digits to be ensured in the approximation of the Kepler's equation solution.*

## INTRODUCTION

The method of successive approximations is one of the powerful tools to solve nonlinear equations numerically.

A remarkable transcendental equation to which one can apply the method of successive approximations is Kepler's equation arising in Astronomy. Kepler's equation gives the relation between the polar coordinates of a celestial body, such as a planet or a comet, and the time elapsed from a given initial point. Kepler's equation is of fundamental importance in Astronomy, but cannot be directly inverted in terms of simple functions in order to determine where the planet will be at a given time.

Kepler’s equation and other nonlinear equations were early solved numerically and intensively studied using the method of successive approximations, as can be seen in (Aaboe A., 2011), (Colwell P., 1993), (Coman G. et al., 1976), (Danby J.M.A., 1988), (Demidowitch B. et al., 1979), (Swerdlow N.M., 2000), but the number of exact digits in these approximations could not be established with certitude when the Lipschitz constant of the involved function is greater than 0.5. In order to improve this status we propose a refinement of successive approximations method by including double or multiple iterations such that the number of exact digits in the approximation to be determined with assuredness. This is important from applications point of view since Kepler’s equation and similar other equations appear in several contexts in applied sciences and engineering.

Moon Margaret of planet Uranus was discovered back in 2001 by S.S. Sheppard and D.C. Jewitt. More recent observations showed it to be a distinct celestial body and on October 9, 2003, its recognition was officially accepted and included in the list as the $27^{th}$ moon of Uranus. The discovery was made using the 8.3 meter Subaru telescope at Mauna Kea (Hawaii). In 2008, Margaret's current eccentricity was 0.7979. This temporarily gives Margaret the most eccentric orbit of any moon in the solar system, though Nereid's mean eccentricity is greater.

## MATERIAL AND METHOD

In the case that the equation:

$$f(x) = 0, x \in R$$

can be transformed into the equivalent form,

$$x = \varphi(x), x \in I \subset R$$

with *I* an interval, the method of successive approximations generates a recurrent sequence:

$$x_{n+1} = \varphi(x_n), n \in N$$

which in certain conditions converges to one of the solution of the equation $x = \varphi(x)$.

When apply the method of successive approximations, each real solution of the equation $x = \varphi(x)$ is isolated in a specified interval by using the method of roots separation generated by the technique of the Rolle's sequence, see (Demidowitch B. et al., 1979). After these, each such interval is shorted as can as possible such that the first derivative $\varphi'$ to satisfy the inequality $|\varphi'(\mathrm{x})| < 1$ on the whole interval. This inequality leads to the convergence of the sequence of successive approximations, given by the recurrence $x_{n+1} = \varphi(x_n), n \in N$, to the solution, starting from any first iterative step $x_0$.

Let *q* be the mean anomaly (a parameterization of time) and *x* the eccentric anomaly (a parameterization of polar angle) of a celestial body (planet or comet) orbiting on an ellipse with eccentricity *E*, then Kepler's equation is given by:

$$x = E \cdot \sin(x) + q \tag{1}$$

where $E \in (0,1]$ and q > 0, see (Aaboe A., 2011) and (Swerdlow N.M., 2000).

In (Aaboe A., 2011), the method of successive approximations is applied to equation (1) for Mars and Mercur with $q = \pi/3$ and $x_0 = 58^{\mathrm{O}}$, while in (Montenbruck O. et al., 2009) the Newton's method is applied to equation (1).

If $x^* \in I$ is the exact solution of the equation $x = \varphi(x)$ and $|\varphi'(\mathrm{x})| < 1, \forall x \in I$, then $\lim_{n \to \infty} x_n = x^*$ and the following a priori error estimate

$$|x^* - x_n| \le \frac{L^n}{1-L}|x_1 - x_0|, \forall n \in N^* \tag{2}$$

and a posteriori error estimate

$$|x^* - x_n| \le \frac{L}{1-L}|x_n - x_{n-1}|, \forall n \in N^* \tag{3}$$

hold, where $\mathrm{L} = \max\{|\varphi'(x)| : x \in I\}, L < 1$ is the Lipschitz constant of $\varphi$, see (Coman G. et al., 1976) and (Demidowitch B. et al., 1979).

The estimate (3) offers a practical stopping criterion of the sequence $x_{n+1} = \varphi(x_n)$, enounced as follows: for given $\varepsilon > 0$, find the first natural number *n* for which $|x_n - x_{n-1}| < \varepsilon$ and stop to this *n*. Then the precision in the approximation of $x^*$ by the term $x_n$ is $|x^* - x_n| \le \frac{L\varepsilon}{1-L}$.

The purpose of this paper is to specify the precision of the approximation by indicating which digits are exact in the decimal representation of $x_n$. How can help the usual method of successive approximations in this aim? For instance, if $L$=0.8, $\varepsilon$ =0.0001=$10^{-4}$ and $x_n$=3.6518, then we have $\left|x^* - x_n\right| < 0.0004$, that is $x^* = x_n \pm 0.0004$. So, $x^* \in (3.6514, 3.6522)$ and we cannot say that the third digit 1 is exact in $x_n$=3.6518. But, if $L \le 0.5$, then $\left|x^* - x_n\right| \le \left|x_n - x_{n-1}\right| < \varepsilon = 10^{-K}$ and all $K$-1 of the first digits in the representation of $x_n$ are exact. More precisely, if $L$=0.5, $x_n$=3.6518, and $\varepsilon = 10^{-4}$, then $\left|x^* - x_n\right| < 0.0001 = 10^{-4}$ and $x^* \in (3.6513, 3.6515)$. Thus, the third digit 1 is exact. From these, we see that if the Lipschitz constant $L$ is such that $L \le 0.5$, then the inequality $\left|x_n - x_{n-1}\right| < \varepsilon$ offers the insurance that the first $K$-1 digits after dot are exact. But if 0.5<$L$<1 , then the inequality $\left|x_n - x_{n-1}\right| < \varepsilon$ not ensure the exactness of the first $K$-1 digits. Can be ameliorated this situation? How? In order to respond to this question we pass to the idea of double iteration and more general, to multiple iteration.

The method of double iteration is described by the sequence

$$x_{n+1} = \varphi(\varphi(x_n)), n \in N \tag{4}$$

<u>Theorem</u>: If $L \le \frac{1}{\sqrt{2}}$ then the a posteriori error estimate $\left|x_n - x_{n-1}\right| < 10^{-K}$ leads to $K$-1 exact first digits after dot, using the method of double iteration (4).

<u>Proof</u>: In the double iteration method, the Lipschitz property

$$\left|\varphi(x) - \varphi(y)\right| \le L\left|x - y\right|, \forall\, x, y \in I$$

leads to

$$\left|x_{n+1} - x_n\right| = \left|\varphi(\varphi(x_n)) - \varphi(\varphi(x_{n-1}))\right| \le L\left|\varphi(x_n) - \varphi(x_{n-1})\right| \le L^2\left|x_n - x_{n-1}\right|, n \in N^*$$

Thus,

$$\begin{aligned}\left|x_{n+p} - x_n\right| &\le \left|x_{n+p} - x_{n+p-1}\right| + \left|x_{n+p-1} - x_{n+p-2}\right| + \ldots + \left|x_{n+1} - x_n\right| \le \\ &\le \left(L^{2p} + L^{2(p-1)} + \ldots + L^2\right)\left|x_n - x_{n-1}\right| = \\ &= L^2 \cdot \frac{1 - \left(L^2\right)^p}{1 - L^2} \cdot \left|x_n - x_{n-1}\right| \le \frac{L^2}{1 - L^2} \cdot \left|x_n - x_{n-1}\right|, \forall n, p \in N^*\end{aligned}$$

and passing to limit for $p \to \infty$, we get:

$$\left|x^* - x_n\right| \le \frac{L^2}{1 - L^2} \cdot \left|x_n - x_{n-1}\right|, \forall n \in N^* \tag{5}$$

which is the a posteriori error estimate corresponding to the double iteration method. If $L \le \frac{1}{\sqrt{2}}$, then $\frac{L^2}{1 - L^2} \le 1$ and $\left|x^* - x_n\right| \le \left|x_n - x_{n-1}\right| < 10^{-K}$ ,$\forall n \in N^*$, that is the precision of $\left|x^* - x_n\right|$ and $\left|x_n - x_{n-1}\right|$ are the same. So, the first $K$-1 digits are exact in the term $x_n$. Since the case $0.5 < L \le \frac{1}{\sqrt{2}}$ extends for $L$ the interval [0 , 0.5] which ensure the exactness of the digits, we infer that the method of double iteration ameliorates the problem of determining the precision when $L$>0.5, but $L \le \frac{1}{\sqrt{2}}$. Consequently, by the method of double iteration, the interval (0 , 0.5] is extended to the interval $\left(0, \frac{1}{\sqrt{2}}\right]$ for the Lipschitz constant $L$, in order to ensure the maximum number of exact digits after dot.

Remark: The method of multiple iteration based on the iterative sequence

$$x_{n+1} = (\underbrace{\varphi \circ \varphi \circ \ldots \circ \varphi}_{p\,times})(x_n), n \in N$$

extend the interval (0 , 0.5] to $\left(0, \frac{1}{\sqrt[p]{2}}\right]$ for the Lipschitz constant, because in this case, the a posteriori error estimate becomes

$$\left|x^* - x_n\right| \le \frac{L^p}{1-L^p} \cdot \left|x_n - x_{n-1}\right|, \forall n \in N^*$$

**NUMERICAL EXPERIMENT**

The experiment tests the method of successive approximations applied to numerical solving of Kepler's equation arising in Astronomy.

Consider Kepler's equation with $E$=1 and $q$=0.25 (corresponding to a comet):

$$x = \sin(x) + 0.25, x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]$$

Here $\varphi(x) = \sin(x) + 0.25$ and

$$L = \max\left\{\left|\varphi'(x)\right| : x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]\right\} = \max\left\{\cos(x) : x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right]\right\} = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}.$$

Thus, the method double iteration is enough to ensure the maximum number of exact digits. Taking $x_0 = \frac{\pi}{4}$ and $\varepsilon = 10^{-4}, 10^{-8}, 10^{-12}$ we obtain with double iterations the results presented in table 1:

*Table 1*

The method of double iteration

| $\varepsilon$ | $n$ | $x_n$ | exact digits |
|---|---|---|---|
| $10^{-4}$ | 6 | 1.17122 | 1.1712 |
| $10^{-8}$ | 11 | 1.1712296516 | 1.17122965 |
| $10^{-12}$ | 16 | 1.1712296525016 | 1.171229652501 |

With simple iteration the results are presented in table 2:

*Table 2*

The method of successive approximations

| $\varepsilon$ | $n$ |
|---|---|
| $10^{-4}$ | 11 |
| $10^{-8}$ | 20 |
| $10^{-12}$ | 30 |

obtaining the same values for $x_n$.

The stopping criterion was: for given $\varepsilon > 0$ find the first natural number *n* for which

$$|x_n - x_{n-1}| < \varepsilon$$

and stop to this iteration *n*, retaining the value $x_n$ as approximation of the solution.

In the case of moon Margaret of planet Uranus, the experimental results obtained for the approximation of Kepler's equation solution with the method of successive approximations and of double iteration are presented in table 3.

As can be viewed in these two examples, we can specify that after 6 double iterations, the approximate of the solution of Kepler's equation applied to parabolic orbits has exact first four digits after dot, and after 10 double iterations, the approximate solution of Kepler's equation applied to the orbit of the moon Margaret has first 20 exact digits after dot. By using the method of successive approximations we need to repeat once the iterations when the sequence becomes stationary, in order to be sure about the number of iterations. So, our proposed method offers the certitude for the number of exact digits.

*Table 3*

Approximation of Kepler's equation solution
for Margaret moon of Uranus planet

| successive approximations | double iteration |
|---|---|
| $\varepsilon$ =1e-20 | $\varepsilon$ =1e-20 |
| n=19 | n=10 |
| $x_0 \div x_{19}$ | $x_0 \div x_{10}$ |
| 0.7853981633974483 | 0.7853981633974483 |
| 1.4012804896930611 | 1.6929876112851803 |
| 1.6929876112851803 | 1.6992433643487772 |
| 1.7001895718653675 | 1.6993536570170995 |
| 1.6992433643487772 | 1.6993556442601510 |
| 1.6993707533921545 | 1.6993556800795808 |
| 1.6993536570170995 | 1.6993556807252190 |
| 1.6993559524325146 | 1.6993556807368566 |
| 1.6993556442601510 | 1.6993556807370664 |
| 1.6993556856343333 | 1.6993556807370702 |
| 1.6993556800795808 | 1.6993556807370702 |
| 1.6993556808253425 | |
| 1.6993556807252190 | |
| 1.6993556807386614 | |
| 1.6993556807368566 | |
| 1.6993556807370989 | |
| 1.6993556807370664 | |
| 1.6993556807370707 | |
| 1.6993556807370702 | |
| 1.6993556807370702 | |

The method exemplified here for Kepler's equation could be applied to other nonlinear equation arising in engineering, physics, and biology, with the same benefit.

## CONCLUSIONS

In order to solve transcendental Kepler's equation arising in Astronomy, one can apply the method of successive approximations which is one of the powerful tools to solve nonlinear equations numerically.

The method of double and multiple iteration could be an improvement of the method of successive approximations in the aim to extend the interval (0,0.5] to $\left(0, \frac{1}{\sqrt[p]{2}}\right]$ for the Lipchitz constant, such that the maximum number of exact digits to be ensured in the approximation of the solution.

This method is a real improvement of the method of successive approximations from numerical point of view.

The applicability of the method consists in the following aspects:

- requires the same conditions as the method of successive approximations;
- in comparison with the method of successive approximations offers the certitude about the number of exact digits of the approximation;
- it is an improvement of the method of successive approximations specifying the necessary number of iterative steps in order to ensure a specified tolerance in the approximation of the solution;
- can be applied to the same examples from the area of sciences and engineering as the method of successive approximations.

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