# TREE TRUNK SHAPE ANALYSIS - CLASSICAL GEOMETRY APPROACH

**Musca Ciprian Dumitru*, Nechita Constantin****

*Stefan cel Mare University of Suceava, Department of Forestry, 13 Universitatii St., 720229, Suceava, Romania, e-mail: ciprianmusca@yahoo.com
**National Institute for Research and Development in Forestry "Marin Drăcea" – INCDS, Calea Bucovinei 73 bis, 725100, Câmpulung Moldovenesc, Romania, e-mail: nechitadendro@gmail.com

***Abstract***
*The form of the trees is a challenge for more than two centuries and yet it remained a current matter that must keep up with technological development. This issue needs answers in real time. To explain and model the taper (tree stem), researchers use various geometrical objects (neiloid, cone, paraboloid, cylinder) and form equations. Solving the problems concerning cone, paraboloid or composite formes is achieving satisfying results using the traditional approach of analytic geometry. In this paper, we have presented, on the basis of data sets, how using classical geometry or equations may create standards that can approximate the taper of trees. The species of trees chosed is spruce, (Picea abies L., H. Karst.) situated in the limit of the ecological species habitat. In the second part of the article we have carried out an experiment through which was tested informations obtained through mathematical modeling of taper form. The results indicate that there are differences between the three different models selected, which under certain conditions can be significant. The environmental factor is the main cause of taper form variability even in restricted areas, sometimes within the same production unit. That induces many iregular shapes which can be retrieved in the proposed models.*

**Key words:** form, taper, shape, volume, errors

## INTRODUCTION

Volume calculation of trees is based on theoretical valuation methods relied on the application of classical geometry elements such as solids of revolution (Behre, 1923). A solid of revolution is defined as a surface rotating around a straight line called pivot (Burkhart, Tome, 2012). The most known methods for calculating the volume that are based on geometric applications of solids of revolution are those proposed by Huber, Newton and Smalian (Giurgiu, 1975; Forslundr, 1982).

Deepening and much better understanding of the classical theory on calculating the volume of trees could help improve understanding the theory of formation of trees taper and by default in identification of easy ways to estimate the volume (Giurgiu, 1979). Determining the volume is often difficult because it depends on the geometry of the taper of the trees (Giurgiu, Drăghiciu, 2005).

The importance of tapper ecuations is relevant in context of multitude of models used (Kozak et al., 1969; Bruce et al., 1968; Kozak, 1988; Newnham, 1988; Cao et al., 1980; Matte, 1949; Tomas, Parkesol, 1991;

Bruce, 1972; Reed, Byrne, 1985; Walters, Hann, 1986; Bi, 2000; Valentine, Gregoire, 2001; Sharma, Zhang, 2004).

## MATERIAL AND METHOD

The material include trees, for which was measured relative square diameters ($d_i^2$ /$D_{bh}^2$) along the stem. Thus $d_i^2$ represent square diameter outside bark at height *i*, and $D_{bh}^2$ is the square diameter at breast height outside bark.

A total of 23 spruce trees were destructively sampled. Diameter at breast height (*d* in centimeters, 1.3 m above ground) was measured to the nearest 0.1 cm to each of stem. The trees were later cut down, leaving stumps of average height 0.3 m and totale bole length was measured to the nearest 0.01 m to estimate the total tree height (in meters). Two perpendicular diameters over bark were measured in ech cross section (at each $h_i$ in meters from ground level), to the diameter <0.1 cm, and were then averaged. The diameters were measured every each meater, resulting a 863 values (Ormerod, 1973).

The volume of the solids of revolution is determined by the function f. The volume may be defined as an object obtained by rotating the subgraph of function f around the pivot. Given a,b ϵ R, a<b and [a, b] → R+ . Set $C_f$ = {(x,y,z) ϵ $R^3 / \sqrt{y^2} + z^2 \leq$ f(x); a≤x≤b} it is called the solid of revolution determined by the function f, or the solid obtained by rotating the subgraph of function f around the pivot Ox.

If function g:[a,b] →$R_+$ is constant on sections, meaning whether there is a division Δ = (a = x0 <.. <xn = b of [a, b]) so that g is constant on each period interval ($x_{i-1}$ , $x_i$), g(x)=$c_i$, any x ϵ ($x_{i-i}$, $x_i$) (xi-1, xi), then the solid of revolution determined by g is a finite cylinder union. The volume of such a solid of revolution is:

$$vol(C_g) = \pi \sum_{i=1}^{n} c_i^2 (x_i - x_{i-1})$$

*The volume of the paraboloid of revolution* is obtained by rotating about the Ox-axis of the function f: [0, b] → R + (Fig. 1).

$$f(x) = \sqrt{2ax} \quad vol(Cf) = \pi \int_0^b f^2(x)dx = 2\pi a \int_0^b xdx = \pi ab^2$$

*The cone volume:* considering: r = 0, the object generated by rotation is the circular cone of height h, the basic circle having radius R (Fig. 1).

$$V = \frac{\pi h R^2}{3}$$

*The volume of the frustum of a cone* is obtained by rotating the trapezoid O'ABO" around the Ox-axis. If we consider r and R as rays of the frustum of cone bases, the equation of the straight-line AB is $y = \frac{R-r}{b-a}(x-a)+r$ where h = b-a represents the height of the cone.

$$vol(Cf) = = \pi\int_a^b \left[\frac{R \quad r}{b \quad a}(x \quad a)+r\right]^2 dx^{x \quad a=t} = \pi\int_a^b \left[\frac{R \quad r}{h}t+r\right]^2 dt = \frac{\pi h}{3}\left[R^2+r^2+Rr\right]$$

Regarding the taper of trees, they can be modeled by division in longitudinal sections and by substituting them through solids of revolution. The general equation has the form:

$D^2 = kx^b$; where according to Graves (1906), b is a constant, $b \geq 0$.

In the cases where b = 0, 1, 2, and 3 are generated by the solids of revolution such as the cylinder, paraboloid, cone, respectively neiloid.

The volume is obtained by integration of equation from above is:

$$V = \int_0^H \pi D^2 dx = \pi k_m \frac{H^{m+1}}{m+1}$$

For cylinder $V = \pi r^2 h$, for cone, that is 1/3 of the volume of the cylinder, $V = \frac{1}{3}\pi r^2 h$, for frustum of cone $V = \frac{1}{3}\pi\left(R^2 + r_2 + Rr\right)h$,

Starting from the relationship V = SL, the volume of the solids of revolution $D^b$, where V is the volume, S the sectional area, L the length of the section, and b, as it was shown before, has the values 0, 1, 2, 3 having as input elements the surface of the section and the length of the piece it results the volumes of the frustum of solids of revolution generated by the general equation.

$V_0 = SL$, volume of the cylinder $V_1 = \left(\frac{S+s}{2}\right)L$, volume of the frustum of paraboloid, mathematical relationship that Smalian used, $V_2 = \left(\frac{S+\sqrt{Ss}+s}{3}\right)L$, volume of the frustum of cone, $V_3 = \left(\frac{S+\sqrt[3]{S^2s}+\sqrt[3]{Ss^2}+s}{4}\right)L$, volume of the frustum of neilod, formula used by Newton and known as *Newton's Prismoid.*

*Formulas of Smalian, Huber and Newton used for volume calculation*. In order to determine the volume of a solid of revolution we consider *l* as the height (length) of the solid of revolution that is divided into sections of length i. The solids thus formed can be assimilated to cylinders with the same volume. The total volume is calculated as the sum of the cylinders volumes. If we consider *S*, $S_{1/2}$ and *s* as the surface areas at the thick end, in the middle, respectively at the small end.

Starting from the relationship V=SL:

*Smalian* assimilates the taper to that of a paraboloid and he proposes the following equation for calculating the volume:

$$V = \frac{1}{2} l(S + s).$$

*Huber* assumes that the average cross-sectional area is at the midpoint of the measured piece. The formula proposed by Huber is:

$$V = \frac{1}{2} l(S_{1/2}).$$

To provide more accuracy, *Newton* developed a formula which is based on measurements at both ends of the measured section and in the middle, as well. Newton proposed formula has the form:

$$V = \frac{1}{6} l(S + 4S_{1/2} + s)$$

*Errors calculation*

Regarding the formulas of Newton and Smalian, the differences can be seen in the relation:

$$(s_m l) - \left(\frac{s_1 + 4s_m + s_2}{6} l\right) = \frac{l}{6}(2s_1 + 2s_2 - 4s_{1/2}) = \frac{l}{6}[2s_{1/2} - (s_1 + s_2)]$$

To analyze how the three formulas, estimate the volume of the taper, we use the method proposed by Assis et al. (2002), based on the elements: bias, SD, SSRR, RP.

$$Bias(D): D = \frac{\Sigma(d_{obs} - d_{estim})}{N},\ SD = \sqrt{\frac{\Sigma((d_{obs} - (d_{estim}) - \bar{D})}{N-1}},$$

$$SSRR = \Sigma\left(\frac{(d_{obs} - d_{estim})}{d_{obs}}\right)^2, \qquad RP = \frac{\Sigma\left(\frac{(d_{obs} - d_{estim})}{d_{obs}}\right)100}{N}.$$

Another method of errors estimation is based on the relation:

$$6V_n = k(D_b^2 + 4\,D_m^2 + D_t^2) = 2V_s + 4V_h\ ;\ 3V_n = V_s + 2V_h\ ;$$

$$V_s = 3V_n - 2V_h\ ; V_s - V_n = E_s\ ; 2V_n - 2V_h = 2E_h\ ; E_s = 2E_h$$

where: $V_n$ is the volume calculated using Newton’s formula, $V_s$ is the volume calculated using Smalian’s formula, $V_h$ is the volume calculated using Huber’s formula, $E_s$ = error ($V_n$-$V_s$) for Smalian’s formula, $E_h$ ¬ = error ($V_n$ - $V_h$) for Huber's formula.

**RESULTS AND DISCUSSION**

The average diameter distribution along the length of the taper and the average tree trunk shape can be observed in Fig. 1. Using formulas of Smalian, Huber and Newton, described above, it was calculated the volume of the 23 trees. The volume was calculated on each segment; felled trees being measured on sections of 1 m length. It was also taken over the volume from the Yield tables, with two entries, d and height, used in Romania.

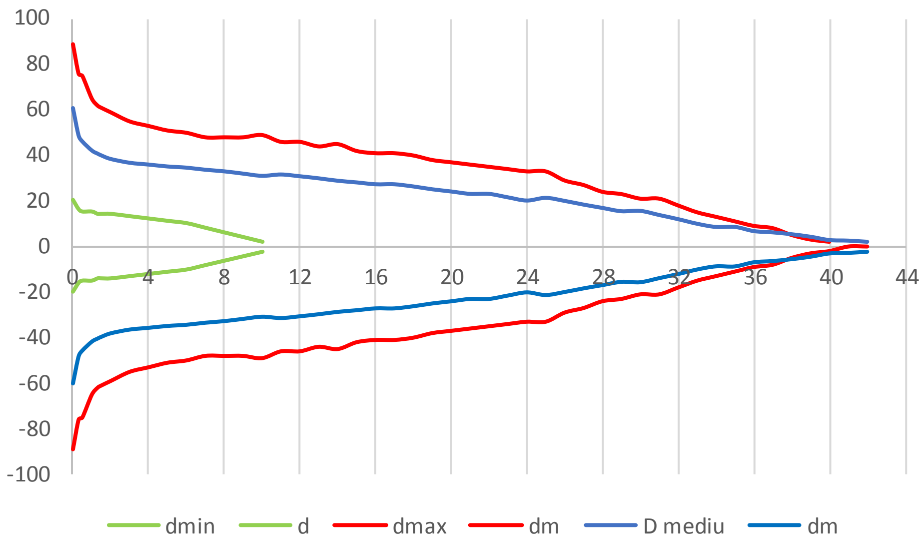


Fig. 1. Diameters distribution relative to the height, profile of the big, small and medium tree

*Table 1*

The volume calculated with the formulas of Smalian, Huber, Newton and the volume extracted from production tables for Romania

| No. Prb. | Volume (mc) | | | |
|---|---|---|---|---|
| | Smalian | Huber | Newton | Yield table |
| 1 | 2,666 | 2,663 | 2,664 | 2,942 |
| 2 | 2,963 | 2,960 | 2,961 | 3,096 |
| 3 | 0,340 | 0,338 | 0,339 | 0,472 |
| 4 | 2,043 | 2,040 | 2,041 | 2,344 |
| 5 | 1,849 | 1,844 | 1,845 | 2,058 |

| No. Prb. | Volume (mc) | | | |
|---|---|---|---|---|
| | Smalian | Huber | Newton | Yield table |
| 6 | 4,888 | 4,886 | 4,887 | 4,464 |
| 7 | 2,993 | 2,989 | 2,991 | 3,307 |
| 8 | 2,681 | 2,676 | 2,678 | 2,714 |
| 9 | 4,701 | 4,696 | 4,698 | 4,591 |
| 10 | 1,654 | 1,650 | 1,651 | 1,558 |
| 11 | 2,472 | 2,470 | 2,471 | 2,419 |
| 12 | 1,189 | 1,185 | 1,186 | 1,230 |
| 13 | 2,616 | 2,613 | 2,614 | 3,070 |
| 14 | 1,352 | 1,348 | 1,349 | 1,404 |
| 15 | 0,095 | 0,095 | 0,095 | 0,079 |
| 16 | 0,773 | 0,771 | 0,772 | 0,681 |
| 17 | 2,300 | 2,298 | 2,298 | 2,365 |
| 18 | 0,684 | 0,682 | 0,683 | 0,681 |
| 19 | 1,955 | 1,953 | 1,954 | 1,828 |
| 20 | 0,487 | 0,486 | 0,486 | 0,449 |
| 21 | 1,596 | 1,593 | 1,594 | 1,555 |
| 22 | 4,677 | 4,672 | 4,674 | 4,708 |
| 23 | 2,158 | 2,155 | 2,156 | 2,193 |
| Vt | 49,131 | 49,064 | 49,086 | 50,208 |

To analyze how the three formulas approximate the shape of the tree from Fig. 1, for sections of 4 and 8 m, the volume is calculated using analyzed formulas. As a reference volume, observed, we will consider the volume calculated using Newton's formula. The data are presented in Table 2. $V_{i=1}$ represents the volume calculated by using Newton's formula on the sections of 1m.

Calculating the errors using we consider the analyzed reference volume as the volume calculated using Newton's formula.

*Table 2*

Errors calculation for the volume achieved by using formulas of Smalian and Huber

| Formula | Nr. tree | V mc | Vmed. | Bias | SD | SSRR | SR | e |
|---|---|---|---|---|---|---|---|---|
| Smalian | 23 | 49,13 | 2,13 | -0,00 | 0,13 | 2,21E-06 | -0,12 | 0,04 |
| Huber | 23 | 49,06 | 2,13 | 0,00 | 0,00 | 1,27E-05 | 0,06 | 0,02 |
| Newton | 23 | 49,08 | 2,13 | | | | | |

To calculate the errors for the sample tree from Fig. 1, the volume observed was considered the resulting volume by applying Newton's formula on sections of 1 m, and in this way we can also follow how Newton's formula approximates the volume on segments of 4 and 8 m.

## CONCLUSIONS

In this article we have analyzed three volume formulas proposed by Huber, Smalian and Newton. Newton proposed formula provides the most correct theoretical calculation of volume samples. Huber's formula, which resembles the taper of the tree to a frustum of paraboloid, may be considered to be the most practical, given that it requires a single diameter measurement on the taper of the section.

Smalian's formula is less accurate than Huber's formula as it uses diameters measured at the two ends of the section excluding the central diameter and in circumstances in which section is very close to the geometric shape of neiloid, this mathematical model gives very large errors. At the same time the calculations show that this formula has a tendency to overestimate the volume.

All three formulas analyzed in this paper provide good results while the length of the pieces whose volume should be calculated is reduced, practically the calculation from this paper was made on segments measured at 1 m distance. Volume calculation methods described in the paper apply specifically to brought down wood to access the elements that must be measured. As appears from the data presented in this paper as the length of the measured section is higher, the accuracy of estimating the volume decreases, the provided formulas failing to capture the natural pattern of the taper.

Newton's proposed formula gives the best results, so it works well for cylinder, neiloid, paraboloid and cone. In practice this method of volume calculation is less used because it requires collecting a significant amount of data.Deepening and much better understanding of the classical theory on volume calculation of trees could help improve understanding the theory of tree taper formation and thus in finding easy ways to estimate the volume.

**Acknowledgments**

This paper has been prepared with the financial support of the project "quality European doctorate - EURODOC", contract no. POSDRU/187/1.5/s/155450, project co-financed by the European social fund through the sectoral operational programme "human resources development" 2007-2013.

**REFERENCES**


1. Behre C.E., 1923, Preliminary Notes on Studies of Tree Form. Journal of Forestry, 21, pp.507- 511
2. Bi H., 2000, Trigonometric variable-form taper equations for australian eucalypts. Forest Science, 46(3), pp.397-409
3. Bruce D., 1972, Notes: Some transformations of the Behre equation of tree form. Forest Science, 18(2), pp.164-166
4. Bruce D., Curtis R.O., Vancoevering C., 1968, Development of a system of taper and volume tables for red alder. Forest Science, 14(3), pp.339-50
5. Burkhart H., Tome M., 2012, Modeling Forest Trees and Stands. Springer
6. Cao Q.V., Burkhart H.E., Max T.A., 1980, Evaluation of two methods for cubic-volume prediction of loblolly pine to any merchantable limit. For. Sci., 26, pp.71-80
7. Forslundr R., 1982, A geometrical tree volume model based on the location of the centre of gravity of the bole. Can. J. For. Res., 12, pp.215-221
8. Giurgiu V., 1975, Expresii matematice ale tabelelor dendrometrice romanesti, Analele ICAS, Vol. 33 (1), pp.75-90
9. Giurgiu V., 1979, Dendrometrie şi auxologie forestieră. Ed. Ceres, Bucureşti
10. Giurgiu V., Drăghiciu D., 2005, Modele matematico-auxologice şi tabele de producţie pentru arborete. Ed. Ceres, Bucureşti
11. Kozak A., 1988, A variable-exponent taper equation. Can. J. For. Res., 18, pp.1363-1368
12. Kozak A., Munro D.O., Smith J.H.G., 1969, Taper functions and their application in forest inventory. For. Chron., 45, pp.278-283
13. Matte L., 1949, The taper of coniferous species with special reference to loblolly pine. The Forestry Chronicle, 25(1), pp.21-31
14. Newnham R.M., 1988, A variable-form taper function. Can. For. Serv. Petawawa Natl. For. Inst. Inf. Rep.PI-X-83
15. Ormerod D.W., 1973, A simple bole model. Forestry Chronicle, v.49, pp.136-138
16. Reed D.D., Byrne J.C., 1985, A simple, variable form volume estimation system. For Chron 61, pp.87-90
17. Sharma M., Zhang S.Y., 2004, Variable-exponent taper equations for jack pine, black spruce, and balsam fir in eastern Canada. Can. J. For. Res., 198, pp.39-53
18. Thomas C.E., Parkesol B.R., 1991, Simple, flexible, trigonometric taper equations. Can. J. For. Res., 21, pp.1132-1137
19. Valentine H.T., Gregoire T.G., 2001, A switching model of bole taper. Can. J. For. Res., 31, pp.1400-1409
20. Walters D.K., Hann D.W., 1986, Taper equations for six conifer species in southwest Oregon. Oregon State University, Forest Research Laboratory, Corvallis, Oregon. Research Bulletin, 56, 36 pp.