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CERTAIN APPLICATION OF DIFFERENTIAL SUBORDINATION

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Abstract
The object of this paper is to derive some inclusion relation regarding a new class of analytic functions using a generalizedRuscheweyhoperator.

Key words:differential subordination, generalized Ruscheweyhoperator,analytic functions.

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INTRODUCTION

1.
In the first section we will recall some definitions and results used for

the new obtained results.

Let $\mathcal{H}$be the class of analytic functions in the open unit disc of the complex plane

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and for $a \in \mathbb{C}$ and n ∈ N let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots,\ z \in U.$$

Let$\mathcal{A}(p,n)$ denote the class of functions $f(z)$ normalized by

(1.1) $f(z) = z^p + \sum_{k=p+n}^{\infty} a_k z^k$, (p, n $\in \mathbb{N} := \{1,2,3,...\}$)

which are analytic in the open unit disc.
In particular, we set

$\mathcal{A}(p,1) := \mathcal{A}_p$ and $\mathcal{A}(1,1) := \mathcal{A} = \mathcal{A}_1$.

Let

$$\mathcal{A}_n = \{f \in \mathcal{H}(U), f(z) = z + a_{n+1}z^{n+1} + \ldots\}$$

with $\mathcal{A}_1 := \mathcal{A}$.

If *f* and *g* are analytic functions in *U*, then we say that function *f* is subordinate to *g* or *g* is said to be superordinate to *f*, if there exists a function *w* analytic in *U*, with $w(0)=0$ and $|w(z)|<1$, and such that $f(z) = g(w(z))$. In such case we write $f \prec g$ or $f(z) \prec g(z)$.

If *g* is univalent, then $f \prec g$ if and only if $f(0)=g(0)$ and $f(U) \subset g(U)$.

Further, we will recall here a differential operator introduced earlier.

Let the function *f* be in the class $\mathcal{A}_n$. For $m, \beta \in N_0 = \{0, 1, 2, \ldots\}$,

$\lambda \geq 0$, $l \geq 0$,

we will use the following differential operator

(1.2) $I^m(\lambda, \beta, l)f(z) := z + \sum_{k=n+1}^{\infty}\left[\frac{1+\lambda(k-1)+l}{1+l}\right]^m C(\beta,k)a_k z^k$

where

$$C(\beta, k) := \binom{k+\beta-1}{\beta} = \frac{(\beta+1)_{k-1}}{(k-1)!}$$

and

$$(a)_n := \begin{cases} 1, & n = 0 \\ a(a+1)\ldots(a+n-1), & n \in N - \{0\} \end{cases}$$

isPochhamer symbol.

Using simple computation one obtains the next result.

MATERIAL AND METHOD

2.PRELIMINARY RESULTS

Proposition 1.1For $m, \beta \in N_0$, $\lambda \geq 0$, $l \geq 0$

(2.1)

$$(l+1)I^{m+1}(\lambda, \beta, l)f(z) = (1-\lambda+l)I^m(\lambda, \beta, l)f(z) + \lambda z(I^m(\lambda, \beta, l)f(z))'$$

and

(2.2)

$$z(I^m(\lambda, \beta, l)f(z))' = (1+\beta)I^m(\lambda, \beta+1, l)f(z) - \beta I^m(\lambda, \beta, l)f(z).$$

Remark 2.1 Special cases of this operator includes the Ruscheweyh derivative operator $I^0(1, \beta, 0)f(z) \equiv D_\beta$ defined in [7], the Sălăgeanderivative operator $I^m(1, 0, 0)f(z) \equiv D^m$, studied in [8], the generalized*Sălăgean operator* $I^m(\lambda, 0, 0) \equiv D_\lambda^m$introduced by Al-Oboudi in [1], thegeneralized Ruscheweyhderivative operator $I^1(\lambda, \beta, 0)f(z) \equiv D_{\lambda,\beta}$introduced in [4], the operator I^m $(\lambda, \beta, 0)$ $\equiv D^m_{\lambda,\beta}$introduced by K. Al-Shaqsiand M. Darus in [2], and the operator $I^m(\lambda, 0, l) \equiv I_1(m, \lambda, l)$introduced in [3].

To prove the main results we will need the following lemma.
Lemma 2.1(Hallenbeck and Ruscheweyh[5]) Let h be a convex function in $U$with $h(0) = a$ and let $\gamma \in C^*$ with Re $\gamma > 0$.
If $p \in \mathcal{H}[a, n]$ and

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec q(z) \prec h(z)$$

where

$$q(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}}} \int_0^z h(t) t^{(\gamma/n)-1} dt.$$

The function q is convex and is the best (a, n)-dominant.

3. MAIN RESULTS

Definition 3.1 Let $f \in \mathcal{A}_n$, $n \in \mathbb{N}^*$. We say that the function f is inthe class $S^m_n(\lambda, l, \alpha, \beta, \eta)$, $\lambda > 0$, $\alpha \in \mathbb{R}$, $\eta \in [0, 1)$, $m \in \mathbb{N}$, if satisfies the condition
(3.1)

$$\mathrm{Re}[(I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))'] > \eta, \quad z \in U,$$

where
(3.2)

$$\widetilde{\Psi}(\alpha, f; z) = z\Psi(\alpha, f; z)$$

and
(3.3)

$$\Psi(\alpha, f; z) = (1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha\left(\frac{zf''(z)}{f'(z)} + 1\right).$$

Theorem 3.1If $\alpha \in \mathbb{R}$, $\eta \in [0, 1)$, $m \in \mathbb{N}$, then
(3.4)

$$S^{m+1}_n(\lambda, l, \alpha, \beta, \eta) \subset S^m_n(\lambda, l, \alpha, \beta, \delta)$$

where
(3.5)

$$\delta = \delta(\lambda, l, \eta, n) = 2\eta - 1 + 2(1 - \eta)\frac{l+1}{n\lambda}\beta\left(\frac{l+1}{n\lambda}\right)$$

and
(3.6)

$$\beta(x) = \int_0^1 \frac{t^{x-1}}{t+1} dt$$

is the Beta function.

Proof.Let $f \in S_n^{m+1}(\lambda, l, \alpha, \beta, \eta)$. By using the properties of the operator$I^m(\lambda, \beta, l)$ we have

$(l + 1)I^{m+1}(\lambda, \beta, l)f(z) = (1 - \lambda + l)I^m(\lambda, \beta, l)f(z) + \lambda z(I^m(\lambda, \beta, l)f(z))'$.

If we denote by

(3.7)

$$p(z) = (I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))'$$

where

$$p(z) = 1 + p_n z^n + \ldots, p(z) \in \mathcal{H}[1, n],$$

thenafter a short computation we get

(3.8)

$$(I^{m+1}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' = p(z) + \lambda z p'(z), z \in U.$$

Since $f \in S_n^{m+1}(\lambda, l, \alpha, \beta, \eta)$, from Definition 3.1 we have

$$\mathrm{Re}(I^{m+1}(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' > \eta, \quad z \in U.$$

Using (3.8) we get

$$\mathrm{Re}(p(z) + \lambda z p'(z)) > \eta$$

which is equivalent to

(3.9)

$$p(z) + \lambda z p'(z) \prec \frac{1 + (2\eta - 1)z}{1 + z} \equiv h(z).$$

From Lemma 2.1 we have

$$p(z) \prec q(z) \prec h(z),$$

where

(3.11)

$$q(z) = \frac{l+1}{n\lambda z^{\frac{l+1}{\lambda n}}} \int_0^z \frac{1 + (2\eta - 1)t}{1 + t} t^{\frac{l+1}{\lambda n} - 1} dt.$$

The function q is convex and is the best $(1, n)$-dominant.

Since

$$(I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))' \prec 2\eta - 1 + \frac{2(1 - \eta)}{n\lambda} \frac{1}{z^{\frac{l+1}{\lambda n}}}, \int_0^z \frac{t^{\frac{l+1}{\lambda n} - 1}}{1 + t} dt, \quad z \in U,$$

it result that

(3.10)

$$\mathrm{Re}[(I^m(\lambda, \beta, l)\widetilde{\Psi}(\alpha, f; z))'] > q(1) = \delta$$

where

(3.11)

$$\delta = \delta(\lambda, l, \eta, n) = 2\eta - 1 + 2(1 - \eta)\frac{l+1}{n\lambda}\beta(\frac{l+1}{n\lambda})$$

and

(3.12)

$$\beta(\frac{l+1}{n\lambda})=\int_0^1 \frac{t^{\frac{l+1}{n\lambda}-1}}{t+1}dt.$$

From (3.10) we deduce that $f \in S_n^m(\lambda, l, \alpha, \beta, \eta)$ and the proof oftheorem is complete. □

Corollary3.1If$\alpha\in\mathbb{R}$, $\eta\in [0, 1)$, $m\in\mathbb{N}$, then
(3.13)

$$S_n^{m+1}(\frac{l+1}{n}, l, \alpha, \beta, \eta) \subset S_n^m(\frac{l+1}{n}, l, \alpha, \beta, \delta)$$

where
(3.14)

$$\delta = \delta(l, \eta, n)= 2\eta - 1 + 2(1 - \eta)ln2.$$

Corollary3.2If$\alpha\in\mathbb{R}$, $\eta\in [0, 1)$, $m\in\mathbb{N}$, then
(3.15)

$$S_n^{m+1}(\frac{1}{n}, 0, \alpha, \beta, \eta) \subset S_n^m(\frac{1}{n}, 0, \alpha, \beta, \delta)$$

where
(3.16)

$$\delta = \delta(\eta, n)= 2\eta - 1 + 2(1 - \eta)ln2.$$

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