# A NOTE ON CERTAIN INEQUALITIES FOR UNIVALENT FUNCTIONS 

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#### Abstract

The concept of uniformly starlike functions and uniformly convex functions were first introduced in [3] by A. W. Goodman and then studied by various authors. In this paper we use a parabolic region to prove certain inequalities for uniformly univalent functions in the open unit disk $U$.


Key words: univalent function, Salagean operator, uniformly starlike, uniformly convex, uniformly close-to-convex

## INTRODUCTION

1. Denote by $U$ the unit disc of the complex plane:

$$
U=\{z \in \mathrm{C}:|z|<1\} .
$$

Let $\boldsymbol{\mathcal { t }}(U)$ be the space of holomorphic functions in $U$ and

$$
\boldsymbol{\mathcal { A }}=\left\{f \in \boldsymbol{\mathcal { H }}(U): f(z)=\mathrm{z}+\mathrm{a}_{2} \mathrm{z}^{2}+\cdots, z \in U\right\} .
$$

Let

$$
S=\{f \in \mathscr{A}: f \text { univalent in } U\}
$$

be the class of holomorphic and univalent functions from the open unit disc $U$.
For $f \in \boldsymbol{\mathcal { A }}, \mathrm{n} \in \mathrm{N}^{*} \cup\{0\}$, let $\mathrm{I} f$ be the Salagean differential operator (see [6]) defined as $\mathrm{I}^{\mathrm{n}}: \boldsymbol{A} \rightarrow \boldsymbol{A}$

$$
\begin{gathered}
\mathrm{I}^{0} f(z)=f(z) \\
\mathrm{I}^{\prime} f(z)=\mathrm{zf} f^{\prime}(z) \\
\ldots \\
\mathrm{I}^{\mathrm{n}+1} f(z)=\mathrm{z}\left[\mathrm{I}^{n} f(z)\right]^{\prime},(z \in U) .
\end{gathered}
$$

## MATERIAL AND METHOD

## 2. Preliminary results

Definition 2.1. A function $f \in \mathrm{~S}$ is said to be in $\operatorname{SP}(\alpha)$, the class of uniformly starlike functions of order $\alpha$, with $\alpha \in[0,1]$, if it satisfies the condition (2.1)

$$
\operatorname{Re}\left\{\frac{z f^{\prime}(z)}{f(z)}-\propto\right\} \geq\left|\frac{z f^{\prime}(z)}{f(z)}-1\right| .
$$

Replacing $f$ in (2.1) by $\mathrm{zf}^{\mathrm{j}}(z)$ we obtain
Definition 2.2. A function $f \in S$ is said to be in the subclass $\operatorname{UCV}(\alpha)$ of uniformly convex functions of order $\alpha$, if it satisfies the condition (2.2)

$$
\operatorname{Re}\left\{1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}-\propto\right\} \geq\left|\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}\right|
$$

The concept of uniformly starlike functions and uniformly convex functions were first introduced in [3] by A. W. Goodman and then studied by various authors.
We set

$$
\Omega_{\alpha}=\left\{u+\imath v, u-\alpha>\sqrt{(u-1)^{2}+v^{2}}\right\}
$$

with

$$
\mathrm{q}(z)=\frac{z f^{\prime}(z)}{f(z)}
$$

or

$$
\mathrm{q}(z)=1+\frac{z f^{\prime \prime}(z)}{f^{\prime}(z)}
$$

and consider the functions which map $U$ onto the parabolic domain $\Omega_{\alpha}$ such that $\mathrm{q}(z) \in \Omega_{\alpha}$. By the properties of the domain $\Omega_{\alpha}$, we have (2.3)

$$
\operatorname{Re}(\mathrm{q}(z))>\operatorname{Re}\left(\mathrm{Q}_{\alpha}(z)\right)>\frac{1+\alpha}{2},
$$

where

$$
\mathrm{Q}_{\alpha}(z)=1+\frac{2(1-\alpha)}{\pi^{2}}\left(\log \frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^{2} .
$$

Furthermore, from [5] we have the following definition
Definition 2.3. A function $f \in \mathrm{~S}$ is said to be in the subclass $\operatorname{UCC}(\alpha)$ of uniformly close-to-convex functions of order $\alpha$, if it satisfies the inequality (2.4)

$$
\operatorname{Re}\left\{\frac{z f^{\prime}(z)}{g(z)}-\propto\right\} \geq\left|\frac{z f^{\prime}(z)}{g(z)}-1\right|
$$

for some $\mathrm{g}(z) \in \operatorname{SP}(\alpha)$.
Remark 2.4. A function $h(\mathrm{z})$ is uniformly convex in $U$ if and only if $\mathrm{z} h^{\prime}(\mathrm{z})$ is uniformly starlike in $U$ (see, for details, [1], [2], [5]).

In order to prove the main results we use the following lemma:
Lemma 2.5. [5] (Jack's Lemma) Let the function $w(\mathrm{z})$ be (non-constant) analytic in $U$ with $w(0)=0$. If $|w(z)|$ attains its maximum value on the circle $|z|=\mathrm{r}<1$ at a point $\mathrm{z}_{0}$, then

$$
z_{0} w^{\prime}\left(z_{0}\right)=c w\left(z_{0}\right),
$$

c is real and $\mathrm{c} \geq 1$.

## 3. Main results

Theorem 3.1. Let $\mathrm{f} \in \boldsymbol{\mathcal { A }}, \mathrm{n} \in \mathrm{N}^{*} \mathrm{U}\{0\}$. If the differential operator $\mathrm{I} n f$ satisfies the following inequality (3.1)

$$
\operatorname{Re}\left(\frac{\frac{n^{n+2} f(z)}{\frac{1}{n+1}(z)^{n^{n+1} f(z)}}}{\frac{1^{n} f(z)}{}}\right)<\frac{5}{3},
$$

then $\mathrm{I}^{\mathrm{n}} f(z)$ is uniformly starlike in $U$.
Proof. We define $w(\mathrm{z})$ by (3.2)

$$
\frac{I^{n+1} f(z)}{I^{n} f(z)}-1=\frac{1}{2} w(z), \quad z \in U .
$$

Then $w(z)$ is analytic in $U$ and $w(0)=0$. Furthermore, by logarithmically differentiating (3.2), we find that

$$
\frac{I^{n+2} f(z)}{I^{n+1} f(z)}-1=\frac{1}{2} w(z)+\frac{z w \prime(z)}{2+w(z)}, \quad z \in U,
$$

which, in view of (3.1), readily yields to (3.3)

$$
\frac{\frac{n^{n+2} f(z)}{n^{n+1} f(z)}-1}{\frac{n^{n+1} f(z)}{1^{n} f(z)}-1}=1+\frac{z w^{\prime}(z)}{\frac{1}{2} w(z)(2+w(z))}, z \in U .
$$

Suppose now, that there exists a point $z_{0} \in U$ such that

$$
\max |w(\mathrm{z})|:|\mathrm{z}| \leq\left|\mathrm{z}_{0}\right|=\left|w\left(\mathrm{z}_{0}\right)\right|=1, \quad\left(\mathrm{w}\left(\mathrm{z}_{0}\right) \neq 1\right)
$$

and, let $w\left(\mathrm{Z}_{0}\right)=\mathrm{e}^{\mathrm{i} \theta},(\theta \neq-\pi)$. Then, applying the Lemma 2.5, we have (3.4)

$$
\mathrm{z}_{0} w^{\prime}\left(\mathrm{z}_{0}\right)=\mathrm{c} w\left(\mathrm{z}_{0}\right), \quad \mathrm{c} \geq 1 .
$$

From (3.3)-(3.4), we obtain

$$
\begin{gathered}
\operatorname{Re}\left(\frac{\frac{n^{n+2} f\left(z_{0}\right)}{\frac{n}{n+1}+1^{n^{n}+1}-1}}{\frac{n^{n+1} f\left(z_{0}\right)}{1^{n} f\left(z_{0}\right)}-1}\right)=\operatorname{Re}\left(1+\frac{z_{0} w \prime\left(z_{0}\right)}{\frac{1}{2} w\left(z_{0}\right)\left(2+w\left(z_{0}\right)\right)}\right)= \\
=\operatorname{Re}\left(1+2 c \frac{1}{2+w\left(z_{0}\right)}\right)=1+2 c \operatorname{Re}\left(\frac{1}{2+w\left(z_{0}\right)}\right)= \\
=1+2 c \operatorname{Re}\left(\frac{1}{2+e^{i \theta}}\right)=(\theta \neq-\pi) \\
=1+2 c \frac{1}{3} \geq 1+\frac{2}{3}=\frac{5}{3}
\end{gathered}
$$

witch contradicts the hypothesis (3.1).
Thus, we conclude that $|\mathrm{w}(z)|<1$ for all $z \in U$; and equation (3.2) yields the inequality

$$
\left|\frac{I^{n+1} f(z)}{I^{n} f(z)}-1\right|<\frac{1}{2}, \quad z \in U
$$

which implies that $\frac{\mathrm{I}^{n+1} f(z)}{\mathrm{I}^{n} f(z)}$ lie in a circle which is centered at 1 and whose radius is $\frac{1}{2}$, which means that $\frac{\mathrm{I}^{n+1} f(z)}{\mathrm{I}^{n} f(z)} \in \Omega_{\alpha}$, and so (3.5)

$$
\operatorname{Re}\left(\frac{\mathrm{I}^{n+1} f(z)}{\mathrm{I}^{n} f(z)}\right) \geq\left|\frac{I^{n+1} f(z)}{\mathrm{I}^{n} f(z)}-1\right|
$$

i.e. In $f(z)$ is uniformly starlike in $U$.

Using (3.5), we introduce a sufficient coefficient bound for uniformly starlike functions in the following theorem:

Theorem 3.2. Let $\mathrm{f} \in \boldsymbol{\mathscr { A }}, \mathrm{n} \in \mathrm{N}^{*} \cup\{0\}$, and the differential operator In $f$. If

$$
\sum_{k=2}^{\infty}(2 k+1-\propto)\left|a_{k+1}\right|<1-\propto
$$

then $\operatorname{In} f(z) \in \operatorname{SP}(\alpha)$.
Proof. Let

$$
\sum_{k=2}^{\infty}(2 k+1-\propto)\left|a_{k+1}\right|<1-\propto .
$$

It is sufficient to show that

$$
\left|\frac{I^{n+1} f(z)}{I^{n} f(z)}-(1-\alpha)\right|<\frac{1+\alpha}{2}, \quad z \in U .
$$

We find that (3.6)

$$
\begin{gathered}
\left|\frac{I^{n+1} f(z)}{I^{n} f(z)}-(1-\alpha)\right|=\left|\frac{-\alpha+\sum_{k-2}^{\infty}(k-\alpha) a_{k+1} z^{k-1}}{1+\sum_{k-2}^{\infty} a_{k+1} z^{k-1}}\right|< \\
<\frac{\alpha+\sum_{k-2}^{\infty}(k-\alpha)\left|a_{k+1}\right|}{1-\sum_{k-2}^{\infty}\left|a_{k+1}\right|}
\end{gathered}
$$

And (3.7)

$$
2 \alpha+\sum_{k=2}^{\infty}(2 k+1-\propto)\left|a_{k+1}\right|<1+\propto .
$$

This shows that the values of the function (3.8)

$$
\Phi(z)=\frac{\mathrm{I}^{n+1} f(z)}{\mathrm{I}^{n} f(z)}
$$

lie in a circle which is centered at $(1+\alpha)$ and whose radius is $\frac{1+\alpha}{2}$, which means that $\frac{\mathrm{I}^{n+1} f(z)}{\mathrm{I}^{n} f(z)} \in \Omega_{\alpha}$. Hence $\mathrm{I}^{n} f(z) \in \operatorname{SP}(\alpha)$.

We determine the sufficient coefficient bound for uniformly convex functions in the next theorem:

Theorem 3.3. Let $\mathrm{f} \in \mathscr{A}, \mathrm{n} \in \mathrm{N}^{*} \cup\{0\}$. If the differential operator $\mathrm{I} f$ satisfies the following inequality (3.9)

$$
\operatorname{Re}\left(\frac{\frac{\frac{I}{}_{n+3}(z)-n^{n+2} f(z)}{n^{n+2} f(z-)^{n+1} f(z)}-2}{\frac{n^{n+2} f(z)}{n^{n+1} f(z)}-1}\right)<3,
$$

then $\operatorname{In}^{\mathrm{n}} f(z)$ is uniformly convex in $U$.
Proof. If we define $w(\mathrm{z})$ by (3.10)

$$
\frac{I^{n+2} f(z)}{I^{n+1} f(z)}-1=\frac{1}{2} w(z), \quad z \in U,
$$

then $w(z)$ satisfies the conditions of Jack's Lemma. Making use of the same technique as in the proof of Theorem 3.1, we can easily get the desired proof of Theorem 3.3.

Theorem 3.4. Let $\mathrm{f} \in \boldsymbol{\mathscr { A }}, \mathrm{n} \in \mathrm{N}^{*} \mathrm{U}\{0\}$, and the differential operator $\mathrm{I}^{\mathrm{n}} f$. If (3.11)

$$
\sum_{k=2}^{\infty}(k+1)(2 k+1-\propto)\left|a_{k+1}\right|<1-\propto,
$$

then $\operatorname{In} f(z) \in U C V(\alpha)$.
Proof. Let

$$
\sum_{k=2}^{\infty}(2 k+1-\propto)\left|a_{k+1}\right|<1-\propto .
$$

It is sufficient to show that

$$
\left|\frac{I^{n+2} f(z)}{I^{n+1} f(z)}-(1-\alpha)\right|<\frac{1+\alpha}{2}, \quad \mathrm{z} \in U .
$$

Making use of the same technique as in the proof of Theorem 3.2, we can prove the inequality (3.11).

The following theorems give the sufficient conditions for uniformly close-to-convex functions.

Theorem 3.5. Let $\mathrm{f} \in \boldsymbol{\mathcal { A }}, \mathrm{n} \in \mathrm{N}^{*} \cup\{0\}$. If the differential operator $\mathrm{I}^{\mathrm{n}} f$ satisfies the following inequality (3.12)

$$
\operatorname{Re}\left(\frac{I^{n+2} f(z)}{I^{n+1} f(z)}-1\right)<\frac{1}{3}
$$

then $\left(\mathrm{I}^{\mathrm{n}} f\right)(z)$ is uniformly close-to-convex in $U$.
Proof. If we define $w(\mathrm{z})$ by (3.13)

$$
\left(\mathrm{I}^{n} f\right)^{\prime}(z)-1=\frac{1}{2} w(z), \quad \mathrm{z} \in U,
$$

then clearly, $\mathrm{w}(\mathrm{z})$ is analytic in $U$ and $\mathrm{w}(0)=0$. Furthermore, by logarithmically differentiating (3.13), we obtain (3.14)

$$
\frac{I^{n+2} f(z)}{I^{n+1} f(z)}-1=\frac{z w \prime(z)}{2+w(z)}, \quad z \in U .
$$

Therefore, by using the conditions of Jack's Lemma and (3.14), we have

$$
\operatorname{Re}\left(\frac{n^{n+2} f\left(z_{0}\right)}{I^{n+1} f\left(z_{0}\right)}-1\right)=c \operatorname{Re}\left(\frac{w\left(z_{0}\right)}{2+w\left(z_{0}\right)}\right)=\frac{c}{3}>\frac{1}{3}
$$

which contradicts the hypotheses (3.12). Thus, we conclude that $|\mathrm{w}(\mathrm{z})|<1$ for all $\mathrm{z} \in \mathrm{U}$; and equation (3.13) yields the inequality

$$
\left|\left(\mathrm{I}^{n} f\right)^{\prime}(z)-1\right|<\frac{1}{2}, \quad \mathrm{z} \in U,
$$

which implies that $\left(\mathrm{I}^{n} f\right)^{\prime}(z) \in \Omega_{\alpha}$, which means

$$
\operatorname{Re}\left(\left(\mathrm{I}^{n} f\right)^{\prime}(z)\right) \geq\left|\left(\mathrm{I}^{n} f\right)^{\prime}(z)-1\right|
$$

And, hence $\left({ }^{\mathrm{I}} \mathrm{f} f\right)(z)$ is uniformly close-to-convex in $U$.

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