# A PARALLEL BETWEEN THE POLLUTANT TRANSPORT EQUATIONS, INTO THE POROUS ENVIRONMENT AND THE BLACK – SCHOLES EQUATION IN EVALUATING STOCK OPTIONS

**Turcan Radu*, Turcan Ciprian Sebastian**, Dumitru Mariana*****

*University of Oradea, Romania, e-mail turcan_radu@yahoo.com
**The Bucharest Academy of Economic Sciences, Romania, ciprian.turcan@gmail.com
***University of Oradea, Romania, e-mail dumitra_mari@yahoo.com

***Abstract***
*In this paper we will briefly present the two main problems regarding: Mathematical equation / model regarding the transport of pollutant into Porous environment (one-dimensional case);*
*Black Scholes model / equation, used to evaluate stocks options.*
*By realizing a side by side analysis of the similitude and differences between the two models, both being expressed as parabolic partial derivatives, very similar one to each other.*

**Keywords** : Model, Stock Options, Portfolio, Pollutant, Porous Environment,.

## INTRODUCTION

In deduction of the partial derivative parabolic equations, used for the two models,

$$\frac{\partial c}{\partial t}+\frac{u}{R}\cdot\frac{\partial c}{\partial x}-\frac{D_h}{R}\cdot\frac{\partial^2 c}{\partial x^2}+\lambda C=0 \qquad (*) \qquad \frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}-rV=0 \qquad (**)$$

we will use the strategies met in the papers [2], [4], and [10], from references. Concretely, we will follow the steps:

S1) We will write the balance equation in the general form (for a fluid with a volume considered to be V)

$$\frac{d}{dt}\left(\int \phi dV\right)+\int \vec{n}\cdot\vec{q}\,dS+\int \sigma dV=0 \quad (1)$$

Where: V= control volume occupied by the fluid
S= the frontier of volume V (surface)
$\phi = C$ property associated to the fluid (concentration)

S'1) We will begin from the variation of the special portfolio, defined as:

$$\pi = V(S;t)-\Delta S \qquad (1')$$

Where $V=V(S;t)$ positive value of an option
S = support asset
ΔS = variation of the support asset value

E'2) We write the relation for the variation of the portfolio:

$d\pi = dV-\Delta dS$ (2′) Where:

**<u>Equations for the Transport of Pollutants Model Black & Scholes Model</u>**

$\vec{q}$ = density of the diffusive / dispersive flows
$\vec{n}$ = exterior normal vector for the surface S
$\sigma$ = No conservative effect density

E2) For the first terms of the equation (1), written above, we will apply:

P1) Transport Theorem for the material derivative , ” $\frac{d}{dt}(\ldots)$, meaning that for the first term of the relation no. (1), we will obtain:

$$\frac{d}{dt}\left(\int \varnothing dV\right) = \int \left(\frac{\partial \varnothing}{\partial t} + \nabla \vec{v} \varnothing\right) dV \quad (2a)$$

P2) We will apply the Divergence Theorem, for the second term from the relation (1), meaning:

$$\int \vec{n} \cdot \vec{q} dS = \int \nabla \vec{q} dV \qquad (2_b)$$

P3) The third term, remains unchanged

E3) We will replace the relations (2a) and (2b) into the relation no (1), and we will obtain:

*dV* = the increasing / variation of the V option

*dS* = the decreasing of the support asset S.

E’3) For the variations: *dV* and *dS*, we will apply Ito’s Lemma (used in the stochastic environment)

P1) For the variation of V, we will apply Ito’s Lemma, and we will obtain:

$$dV = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \qquad (3')$$

(for the increasing of V)

and $dS = \mu S dt + \sigma S dX$ (For the increasing of S) (3˝)

Obs! Here, we will simply write:

$dS = \mu S dt + \sigma S dX$ (evidently, true) (3˝)

We will replace the *dV* expression, given by the relation (3’), into the relation (2’), we will obtain:

$$\int \left[\left(\frac{\partial \varnothing}{\partial t} + \nabla \vec{v} \varnothing\right) + \nabla \vec{q} + \sigma\right] dV = 0 \qquad (3)$$

E4) Taking in consideration that V = arbitrary volume, by using the “Null Integral Lemma” in the relation no (3), written above, we will obtain:

$$\frac{\partial \varnothing}{\partial t} + \nabla \vec{v} \varnothing + \nabla \vec{q} + \sigma = 0 \qquad (4)$$

(local form of the balance equation, for the property $\varnothing$)

E5) We replace the property $\varnothing$ from relation no (4), with C, meaning: $\varnothing$ = C (pollutant concentration) and then the relation no. (4) will become:

$$\frac{\partial C}{\partial t} + \frac{\vec{u} \nabla C}{R} - \frac{1}{R} \nabla (D \cdot \nabla C) + \lambda C = 0 \qquad (5)$$

(the standard case, for the general equation of the pollution transport into porous environment).

E6) We will reduce the dimension of the equation no (5), from the case D3 (tridimensional case) to the case D1 (the one-dimensional case), and we

$$d\pi = \frac{\partial V}{\partial t} dt + \frac{\partial V}{\partial S} dS + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \Delta dS \ (4')$$

E5) We will write under equivalent form, the relation no (4’) and we will have:

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt + \left(\frac{\partial V}{\partial S} - \Delta\right) dS \ (5')$$

E6) We will apply the hedging strategy (the eliminating risk strategy) by imposing condition:

$$\frac{\partial V}{\partial S} - \Delta = 0 \qquad (6')$$

E7’) We replace (6’) to (5’), and we will obtain:

$$d\pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt \qquad (7')$$

(meaning: the variation of portfolio , risk free)

E8') The relation (7'), written above, has the

Financial equivalent: $d\pi = r\pi dt$ (8')

(Meaning the equivalent to cash, from a bank deposit, with an interest r, free risk).

E9') We will use the Non-Arbitrage Principle", writing the equivalent of relations (7') and (8'), meaning:

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt = r\pi dt \;\Big|\; : dt \neq 0 \quad (9')$$

Will finally obtain the asked question: $\frac{\partial c}{\partial t} + \frac{u}{R}\cdot\frac{\partial c}{\partial x} - \frac{D_L}{R}\cdot\frac{\partial^2 c}{\partial x^2} + \lambda C = 0$ qed (6)

(Pollution transport equations into the porous environment – one-dimensional case)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\pi \qquad (9'')$$

After the simplification with the quantity $dt \neq 0$, The relation no (9'), becomes:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\pi \qquad (9'')$$

$$\left.\begin{array}{l}\text{but}: \pi = V - \Delta S \\ \text{and } \Delta = \frac{\partial V}{\partial S}\end{array}\right\} \Rightarrow \pi = V - S\frac{\partial V}{\partial S} \qquad (9''')$$

E10') We will replace (9''') -> (9''), and we will obtain:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = r\left(V - S\frac{\partial V}{\partial S}\right) \qquad (10')$$

Or equivalent …

$$\boxed{\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0} \qquad (10'')$$

## RESULTS AND DISCUSSION

***The Black & Scholes Equation***

**The interpretation of the terms from the equations (*) and (**), for the two models M1 and M2 (equations written above).**

**Equations for the Transport of Pollutants Model Black & Scholes Model**

$$\frac{\partial c}{\partial t} + \frac{u}{R}\cdot\frac{\partial c}{\partial x} - \frac{D_L}{R}\cdot\frac{\partial^2 c}{\partial x^2} + \lambda C = 0 \qquad (*)$$

Where:

$C = C(x, t)$ = concentration of pollutant

$u$ = real speed (in module) of the fluid in the porous environment

$R$ = "being late" factor for the pollutant

$D_L$ = Longitudinal dispersion coefficient

$\lambda C$ Decomposing rate (speed) of the pollutant

$\frac{\partial c}{\partial t}$ → local accumulation component

$\frac{u}{R}\cdot\frac{\partial c}{\partial x}$ → Convective transport component

$\frac{D_L}{R}\cdot\frac{\partial^2 c}{\partial x^2}$ → Dispersive/diffusive transport component

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \qquad (**)$$

Where:

$S = S(\mu; \sigma) \rightarrow$ Value of the support asset (stochastic variable)

$V = V(S; t) \rightarrow$ Call options value

$r$ = Interest rate

$t$ = Time (determinist value) And

$\frac{\partial V}{\partial t} = \theta$ → Theta indicator represent, the valuation of the option price V depending by the time variation t.

$\frac{\partial V}{\partial S} = \Delta \rightarrow$ Delta indicator – represent: delta hedging (good instrument in eliminating risk)

$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\partial V}{\partial S}\right) = \Gamma$ → Gamma indicator – represent the excessive sensitivity of the delta $\frac{\partial V}{\partial S} = \Delta$, to price variation of the support asset S.

**METHODS**

**Methods of resolving the equations (*) and (**)**

**Equations for the Transport of Pollutants Model Black&Scholes Model**

In general, the resolving methods for the pollution transport equation (one-dimensional case), may be classified in: analytical methods and numerical methods

$$\frac{\partial C}{\partial t} + \frac{u}{R} \cdot \frac{\partial C}{\partial x} - \frac{D_L}{R} \cdot \frac{\partial^2 C}{\partial x^2} + \lambda C = 0 \qquad (*)$$

Analytical methods may be exactly or approximately.

There are some numerical methods as:

MEDIF – Finite Difference Method

MEVFIN –Finite Volume Method

MEFIN – Finite Elements Method

MEFRO – Frontier Elements Method

Next, we will present an analytical solution, for the equation (*)

Black & Scholes Equations purpose consist in finding the exactly solution for the parabolic partial derivatives equation.

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \qquad (**)$$

There are two main methods mentioned in the specialty literature, in order to resolve this type of equation: Analytical methods and Numeric methods.

As analytical methods, we may remind: a method consisting in transforming the equation (**), into another equation with constant diffusion coefficient, equation obtained by changing the variable.

**Equations for the Transport of Pollutants Model Black & Scholes Model**

In this sense, we will take in consideration the following aspects:

Characteristics / parameters

One dimensional aquifer (1D / D1) infinite, meaning: $C = C(x; t)$; $t \in (-\infty; \infty)$

Transport: convective, dispersive/diffusive and degradation

Instantaneous injection

One dimensional speed field

Main equation

$$\frac{\partial C}{\partial t} + \frac{V_x}{R} \cdot \frac{\partial C}{\partial x} - \frac{D_L}{R} \cdot \frac{\partial^2 C}{\partial x^2} + \lambda C = 0$$

Initial conditions:

$C(x; t=0) = \frac{M_0}{mnR}\delta(x-0)$ and limit conditions

Analytical solutions

$C(x;t) = \frac{M_0}{2mnR\sqrt{\pi \cdot D \cdot t/R}} e^{\left[-\frac{(x-v_x \cdot t/R)^2}{4Dt/R} - \lambda t\right]}$ with $C_{max} = \frac{M_0}{mnR}$; Where:

$M_0$= injection pollution amount to the source

$\delta(x-0)$ = Delta Dirac Function

m= the thickness of the aquifer

D= dispersion / diffusion coefficient

$C_{max}$ = Maximum concentration, initial to source.

5) For the analytical solution,

$V(S;t) = e^{\alpha x + \beta \tau} \cdot \mathcal{U}(x;\tau)$.

Finally, the function (solution) $\mathcal{U}(x;\tau)$ satisfies the following equation:

$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$ (main equation for the molecular diffusion). Observation!

O1) The equation written above, $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$, is easier to resolve it, than the initial equation (**) (Black Scholes)

O2) As the numerical methods utilized in resolving the Black & Scholes equation (**), we may remind:

Finite differential methods, with its variants:

Explicit finite differential Method

Implicit finite differential method

O3) Very well known, is the Crank – Nicolson Method (numerical methods), method that realizes: “the average between the explicit and implicit method”.

04) A particular solution, for the diffusion equation, $\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$, has got the following form:

$$W(x;\tau) = \frac{1}{\sqrt{2\pi\sigma\tau}} e^{-\frac{(x-x')^2}{2\sigma^2\tau}}$$

$$C(x;t) = \frac{M_0}{2mnR\sqrt{\pi \cdot D \cdot \frac{t}{R}}} e^{\left[-\frac{\left(x - v_x \cdot \frac{t}{R}\right)^2}{\frac{4Dt}{R}} - \lambda t\right]}$$

we have the following graphical / intuitive interpretation

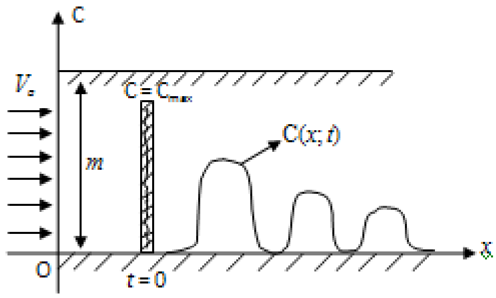


Figure 1: Graphical interpretation for the analytical solution.

Relation that represent: "The probability of the density function for a random variable, normally distributed with zero average and standard deviation $\sigma$"

O5) As a intuitive aspect, the particular solution $W(x;\tau)$,written above, may be represented in the Cartesian system $Ox\tau$, as:

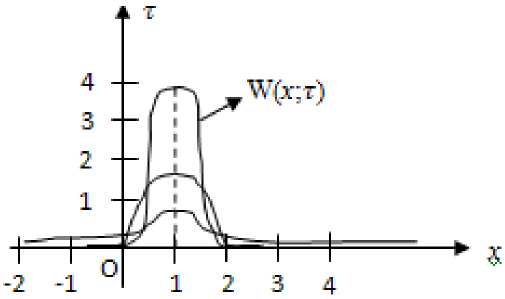


Figure 2: Graphical interpretation of the particular solution $W(x;\tau)$

**4. Other similar models to the M1 and M2**

The two equations:

$\frac{\partial C}{\partial t} + \frac{u}{n} \cdot \frac{\partial C}{\partial x} - \frac{D_L}{n} \cdot \frac{\partial^2 C}{\partial x^2} + \lambda C = 0$ (*) and $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$ (**),

Deducted equations and resolved before, represent mathematical models for:

The transport of pollutants into the porous environments (the one dimensional case) and

The Black – Scholes Model, used to evaluate stocks options.

These models described above, are similar to the next ones:

**M3) Molecular Diffusion Equation Model**

General Form:

$\frac{\partial C}{\partial t} = D \cdot \nabla^2 C \quad \Leftrightarrow \quad \frac{\partial C}{\partial t} = D \cdot \Delta C$

Where: $C = C(x;y;z;t)$ = the concentration of solution

$D$ = diffusion coefficient ($D = D$ 2nd order tensor)

$\nabla^2 C \stackrel{\text{def}}{=} \Delta C = \frac{\partial^2 C}{\partial x^2} + \frac{\partial^2 C}{\partial y^2} + \frac{\partial^2 C}{\partial z^2}$ (C' laplacean)

$\frac{\partial C}{\partial t}$ = gradient (the increases) of the C's concentration, over time:

Particular form: $\frac{\partial C}{\partial t} = D_L \cdot \frac{\partial^2 C}{\partial x^2}$ (diffusion equation for the case 1D)

**M4) The Heat Equation:**

$$\frac{\partial U}{\partial t} = k \cdot \frac{\partial^2 U}{\partial x^2};$$

Where: $\mathcal{U}$ = heat / temperature

$x$ = spatial coordinate (x axis)

$t$ = temporal coordinate (t=time)

$\frac{\partial U}{\partial t}$ = gradient ("the increase") of the temperature:

$\frac{\partial^2 U}{\partial x^2}$=the heat stored in a paralepipedic bar

$k$ = coefficient (for the heat equation)

## CONCLUSIONS

Equation: $\frac{\partial C}{\partial t} + \frac{u}{R} \cdot \frac{\partial C}{\partial x} - \frac{D_L}{R} \cdot \frac{\partial^2 C}{\partial x^2} + \lambda C = 0$ (*)

the mathematical model for pollutants transport in porous environment (case 1D), has been studied and analyzed by foreign and local scientists from fluid mechanics field, having representative papers (papers reminded to references)

The Equation: $\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV = 0$ (**) represents the Black Scholes Model, used in the evaluation of stocks options, and has been written for the firs time, under this form in 1969, but published only in 1973, when Fischer Black and Myron Scholes have demonstrated the fairness of this equation.

A lot of foreign and Romanian specialists have been preoccupied in this field (papers reminded also at references).

The two models/equations, presented above, are very similar to each other, both of them representing: parabolic partial derivatives equation, equations that model a special equation from the "Fluids Mechanics" of the type: reaction, convection, diffusion, as it result from this paper.

## REFERENCES


1. Bear, J., 1979, Hydraulics of Groundwater, Mc – Graw Hill, New York
2. Black, F., M, Scholes, 1973, The pricing of options and corporate liabilites, Journal of Political Economy
3. Bredehoef, J. D., 1973, Mass Transport in Flowing Groundwater, Water Resources Research, Vol. 9, No. , 1973
4. David, I., I., Şumălan, , A. Nituşcă, 1996, Transportul poluanţilor prin medii poroase, Editura Universităţii „Politehnica" din Timişoara,
5. David, I., 1998, Groundwasserhydraulik, Stromung-und Transportvorgänge, Braunschweig, wiesbaden, Vieweg
6. Hull, J., 2008, Options, Futures and other derivatives Securities, Pearson Prentice Hall
7. Negrea, B, 2006, Evaluarea activelor financiare Editura „ECONOMICA", Bucureşti
8. Stancu I. (coord), 2004, „Finanţe. Pieţe financiare şi gestiunea portofoliului", Vol. I, Editura „ECONOMICA", Bucureşti
9. Turcan, R, 2010, Black – Scholes Model Used to Evaluate Stock Options, International Conference „European Integrations,New Challenges for the Romanian Economy",6th Edition Oradea
10. Wilmott, P., 2002, Derivatives – The Theory and Practice of Financial Engineering", University Edition, London